A degenerate double-zero bifurcation in a normal form of Lorenz's equations

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Abstract. In this work we consider an unfolding of a normal form of the Lorenz system near a triple-zero singularity. We are interested in the analysis of a double-zero bifurcation emerging from that singularity. The local study of the double-zero bifurcation provides partial results that are extended by means of numerical continuation methods. Specifically, a curve of heteroclinic connections is detected. It has a degenerate point from which infinitely many homoclinic connections emerge. In this way, we can partially understand the dynamics near the triple-zero singularity.

Introduction

We consider a three-parameter unfolding, that is close to the normal form of the triple-zero bifurcation exhibited by the Lorenz system, given by

$$\dot{x} = y, \qquad \dot{y} = \epsilon_1 x + \epsilon_2 y + Axz + Byz, \qquad \dot{z} = \epsilon_3 z + Cx^2 + Dz^2, \tag{1}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \approx 0$ and A, B, C, D are real parameters. System (1) exhibits a triple-zero bifurcation when $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$. These equations are also invariant under the change $(x, y, z) \to (-x, -y, z)$.
We remark that several systems studied in the literature appear as particular cases of (1) for certain parameter choices: the Shimizu-Morioka system [1, 2] and a low-order model of magnetoconvection [3]; a Lorenz-like system [4, 5]. Moreover, system [6, Eq. (2.7)], under certain conditions, has non-degenerate heteroclinic cycles that connect the equilibria located on the z-axis.
In our case, if $AC \neq 0$, we take without loss of generality $A = -1, C = 1$:

$$\dot{x} = y, \qquad \dot{y} = \epsilon_1 x + \epsilon_2 y - xz + Byz, \qquad \dot{z} = \epsilon_3 z + x^2 + Dz^2. \tag{2}$$

System (2) can have up to four equilibria, namely $E_1 = (0,0,0)$, $E_2 = (0, 0, -\epsilon_3/D)$ if $D \neq 0$ and $E_{3,4} = \left(\pm\sqrt{-\epsilon_1(\epsilon_3 + D\epsilon_1)}, 0, \epsilon_1\right)$ if $\epsilon_1(\epsilon_3 + D\epsilon_1) < 0$. Note that E_1 and E_2 are placed on the z-axis, that is an invariant set. Our goal is the analysis of the double-zero bifurcation exhibited by the equilibrium $E_1 = (0, 0, 0)$, when $(\epsilon_1, \epsilon_3) = (0, 0)$, $\epsilon_2 \neq 0$, and its degeneracies.

Results and discussion

By means of numerical continuation methods, the local results can be extended and applied to the study of (2) when $B < 0$ and $D > 0$. In this way, we can partially understand the dynamics around the triple-zero singularity. We detect non-degenerate heteroclinic cycles, for small values of ϵ_1 and ϵ_3 (when $\epsilon_2 \neq 0$), that are related to a double-zero degeneracy undergone by the origin when $(\epsilon_1, \epsilon_3) = (0, 0)$. Furthermore, near the triple-zero singularity, the heteroclinic connection becomes degenerate. This fact, among other global connections, gives rise to infinite homoclinic orbits that will lead to the existence of chaos (see Fig. 1).

References

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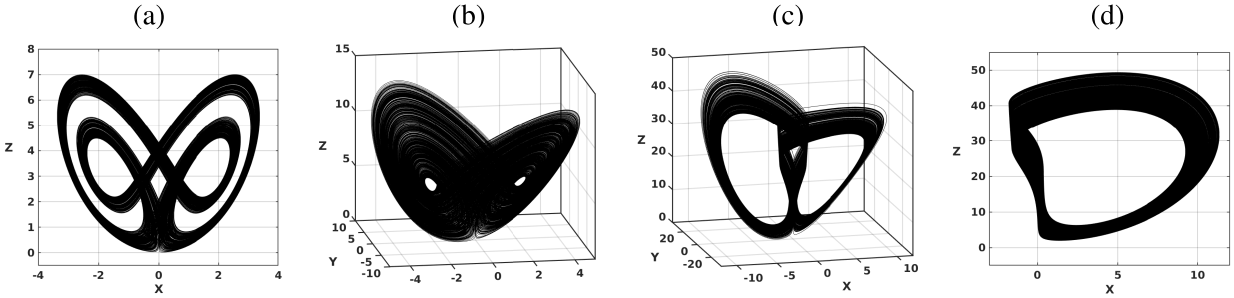

Figure 1: For $\epsilon_2 = -1, \epsilon_3 = -1, B = -0.1, D = 0.01$, geometric Lorenz attractors in system (2) when: (a) $\epsilon_1 = 3$; (b) $\epsilon_1 = 5$; (c) $\epsilon_1 = 15$; (d) $\epsilon_1 = 16.3$.