Stability boundaries for generic two-step Hill's equations

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Abstract. Stability boundaries in a three-parameter space for Hill's equation with two-step periodic potentials are studied. Their geometric structure is described completely in the elliptic case, by analyzing the monodromy matrix of the system and introducing new suitable parameters. In particular, we show that there exist some families of curves where the monodromy matrix is plus or minus the identity, which organize the whole bifurcation set.

Introduction

For the Hill's equation with a two-step piecewise constant function as T-periodic potential, namely

$$\ddot{x}(t) + V(t)x(t) = 0, \quad V(t) = \begin{cases} a, & 0 \le t \le \tau, \\ b, & \tau \le t \le T, \end{cases} \tag{1}$$

we study the elliptic case ($a, b > 0$). This case, considered in [3], is a generalization of the so-called Meissner equation, see [4]. Other works related to the geometrical aspects of the resonance region for equation (1), see [1, 2], assumed for the potential V to be a square wave ($T = 2\tau$). Letting $X(T)$ be the monodromy matrix of Eq. (1), our goal is to describe the parametric stability boundaries given by the surfaces $\operatorname{trace} X(T) = \pm 2$.

Main results

Proposition 1 *Take $\omega_a, \omega_b > 0$ with $a = \omega_a^2$, $b = \omega_b^2$, $T = 2\pi$ and $\tau = rT$ for $(\omega_a, \omega_b, r) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (0, 1)$. For the points belonging to the two denumerable families of curves*

$$\Gamma_{k,m} = \left\{ (\omega_a, \omega_b, r) : \omega_a = \frac{k}{2r},\ \omega_b = \frac{m}{2(1-r)},\ r \in (0,1) \right\}, \Sigma_n = \left\{ (\omega_a, \omega_b, r) : \omega_a = \omega_b = \frac{n}{2}, r \in (0,1) \right\},$$

the monodromy matrix of Eq. (1) satisfies $X(T) = I$ if $k + m$ or n is even while $X(T) = -I$ if $k + m$ or n is odd. The points $P_{k,m}$ of $\Gamma_{k,m}$ with $(k + m)r = k$ are also in Σ_n for $n = k + m$, and constitute the highest co-dimension points of the bifurcation set.

Our key result arises after introducing new suitable parameters, as are the *mean* frequency ϖ, the *relative separation* between frequencies ϱ, and their global *discrepancy* δ, namely

$$\varpi = r\omega_a + (1 - r)\omega_b > 0, \quad \varrho = \frac{\omega_a - \omega_b}{\omega_a + \omega_b} \in (-1, 1), \quad \delta = r\omega_a - (1 - r)\omega_b \in (-\varpi, \varpi). \tag{2}$$

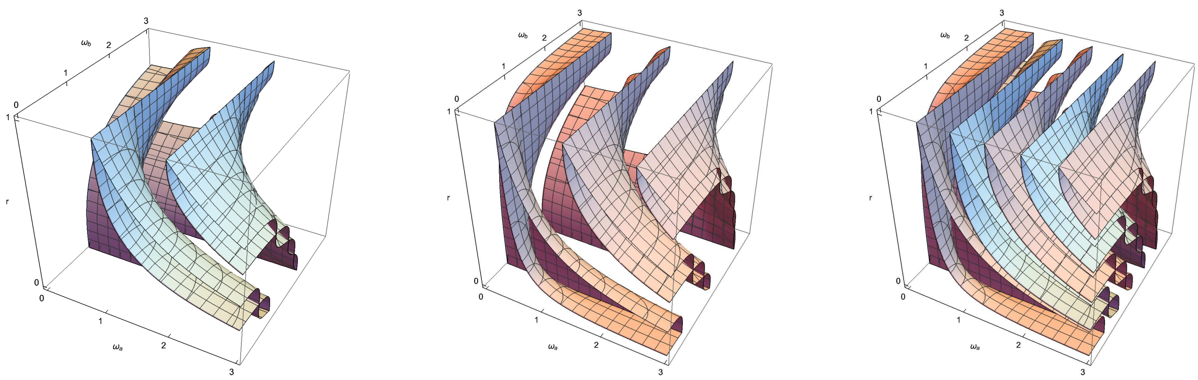

Figure 1: The surfaces corresponding to $\operatorname{trace} X(2\pi) = 2$ (light blue), $\operatorname{trace} X(2\pi) = -2$ (light red) and all of them.

Theorem 1 *In terms of the new parameters given in (2), the analytical expressions for the parametric surfaces $\operatorname{trace} X(2\pi) = 2$ and $\operatorname{trace} X(2\pi) = -2$ (the stability boundaries) are given by $\sin^2(\pi\varpi) - \varrho^2 \sin^2(\pi\delta) = 0$, and $\cos^2(\pi\varpi) - \varrho^2 \cos^2(\pi\delta) = 0$, respectively. Coming back to the original parameters, you can describe completely the stability boundaries that determine the resonance pockets, see Figure 1.*

References

[1] Broer H., Levi M. (1995). Geometrical Aspects of Stability Theory for Hill's Equations, *Arch. Rational Mech. Anal.* **131**: 225-240.

[2] Broer H., Simó C. (2000). Resonance Tongues in Hill's Equations: A Geometric Approach, *Journal of Differential Equations.* **166**: 290-327.

[3] Gan S., Zhang M. (2000). Resonance pockets of Hill's equations with two-step potentials, *SIAM J. Math. Anal.,* **32-3**:651-664.

[4] Magnus W., Winkler S. (1966), *Hill's Equations*, Wiley, New York.