KdV, extended KdV, 5th-order KdV and Gardner equations generalized for uneven bottom versus corresponding Boussinesq's equations

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Abstract. The main goal of the presentation is to compare the motion of solitary surface waves resulting from two similar but slightly different approaches. In the first approach, the numerical evolution of soliton surface waves moving over the uneven bottom is obtained using single wave equations. In the second approach, the numerical evolution of the same initial conditions is obtained by the solution of a coupled set of the Boussinesq equations for the same system of the Euler equations. We discuss four physically relevant cases of relationships between small parameters α, β, δ. For the flat bottom, these cases imply the Korteweg-de Vries equation (KdV), the extended KdV (KdV2), fifth-order KdV (KdV5), and the Gardner equation.

Introduction

In [1], we derived four KdV-type nonlinear wave equations that generalize already known equations to the case of the uneven bottom. Besides standard small parameters $\alpha = \frac{a}{H}$ and $\beta = \left(\frac{H}{l}\right)^2$ we introduced the third one, defined as $\delta = \frac{a_h}{H}$. Here a represents the wave amplitude, H - average depth of the basin, l - average wavelength and a_h - amplitude of bottom variations. For the perturbation approach, all of them should be small but not necessarily of the same order. Therefore for the different ordering of these parameters, one can derive different sets of the Boussinesq equations and, consequently, different wave equations. Surprisingly, the new term in wave equations has, in all considered cases, the same, universal form $Q_{\text{bott}} = -\frac{1}{4}\delta(2h\eta_x + h_x\eta)$. The Boussinesq sets of equations can be made compatible (in other words, can be reduced to a single wave equations) only when $\frac{d^2h}{dx^2} = 0$. So, we obtained the following equations (all written dimensionless variables, in fixed reference frame) generalized for an uneven bottom:

- Korteweg-de Vries equation (KdV), $\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{3x} + Q_{\text{bott}} = 0;$
- *Extended* Korteweg-de Vries equation (KdV2),
$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{3x} - \frac{3}{8}\alpha^2\eta^2\eta_x + \alpha\beta\left(\frac{23}{24}\eta_x\eta_{2x} + \frac{5}{12}\eta\eta_{3x}\right) + \frac{19}{360}\beta^2\eta_{5x} + Q_{\text{bott}} = 0;$
- 5th-order Korteweg-de Vries equation (KdV5),
$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1-3\tau}{6}\beta\eta_{3x} + \frac{19-30\tau-45\tau^2}{360}\beta^2\eta_{5x} + Q_{\text{bott}} = 0;$
- Gardner equation, $\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{3}{8}\alpha^2\eta^2\eta_x + \frac{1-3\tau}{6}\beta\,\eta_{3x} + Q_{\text{bott}} = 0.$

Results and discussion

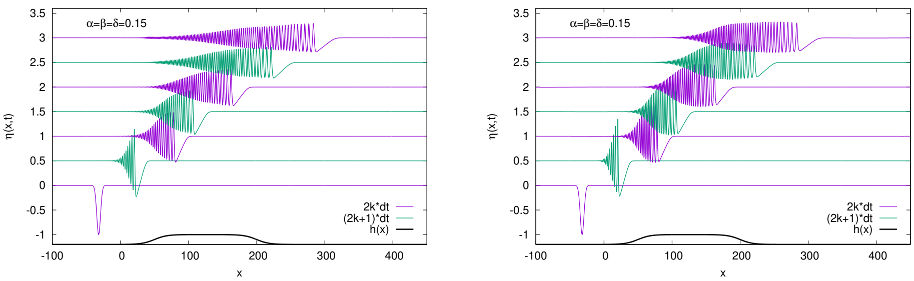

Figure 1: Left part - time evolution obtained according to the generalized KdV equation. Initial Gaussian profile with the triple volume of the KdV soliton, the same velocity, but the inverse amplitude. Here, time step between the consecutive profiles is $dt = 64$. Right part - the same obtained from the corresponding Boussinesq's equations. Black line markes the shape of the bottom, not in scale.

Extensive numerical studies of all four equations are presented in [2]. In each case, we also performed the calculations according to the corresponding Boussinesq's equations and compared the obtained motion of the surface wave. The main property of the results is that the influence of the uneven bottom on the surface wave $\eta(x,t)$ obtained from the Boussinesq equations is always substantially greater than that obtained from single KdV-type wave equations. The example of such properties is presented in Fig. 1.

References

[1] Karczewska, A., Rozmej, P. (2020) Can simple KdV-type equations be derived for shallow water problem with bottom bathymetry? *Commun Nonlinear Sci Numer Simulat* **82**:105073.

[2] Karczewska, A., Rozmej, P. (2020) Generalized KdV-type equations versus Boussinesq's equations for uneven bottom – numerical study. arXiv:2007.01267.