

# Universal upper estimate for prediction errors under moderate model uncertainty

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**Abstract.** We present a method of sensitivity analysis for general dynamical systems, subjected to deterministic or stochastic modeling uncertainty. Using the properties of the unperturbed dynamics, we derive a universal bound for the leading-order prediction error. This bound motivates the definition of the Model Sensitivity, a scalar quantity, depending on the initial condition and time. We demonstrate, using nonlinear numerical models that the Model Sensitivity provides both a global view over the phase space of the dynamical system and in some situations, a localized, time-dependent predictor of uncertainties along trajectories. We find that the phase-space structure of the Model Sensitivity (MS) is related, but not identical to that of the Finite-Time Lyapunov Exponents (FTLE). We formulate conditions under which robust features of the FTLE field are expected to also be seen in the MS field.

## Introduction

In modeling real-world phenomena uncertainties often enter the model formulation, arising as a result of noisy data or uncertainty in the mathematical model. A related concept is sensitivity analysis, which assesses the the response of the model to infinitesimal changes of initial data or parameters. Typically, the estimation of sensitivities and prediction errors requires the use of Monte Carlo methods. In addition to being computationally intensive, one has to make assumptions on the nature of the present uncertainties. For specific classes of systems, efficient methods are available for calculating sensitivities. For example, linear response theory [1] and the least squares shadowing method [2] proved to be effective in computing sensitivities characterizing infinite time averages of observable quantities. Our recent work [3] shows that for finite times, one is able to bound the impact of these uncertainties on model trajectories. The bounds utilize minimal information on the modeling errors but substantial information on the internal dynamics of the known model.

## Results and discussion

We discuss a set-up in which the idealized (nonlinear) model is assumed to be known in the form of an ordinary differential equation, to which model uncertainty is added as a perturbation with bounded amplitude, which can have both a deterministic and a stochastic source. We derive an upper estimate on the leading-order mean-squared prediction error that is valid along trajectories of the idealized model. We also introduce the Model Sensitivity (MS) as the coefficient relating the prediction error to the model uncertainty. Being a function of time and the initial condition of the model trajectory, MS provides an assessment of regions of phase space exhibiting high sensitivity to model uncertainties. Figure 1 shows an example of a representation of MS as well as the time dependence of the leading-order mean-squared prediction error along a model trajectory. We examine the expectation that Finite-Time Lyapunov Exponents (FTLE) capture sensitivity to modeling errors by relating the arising MS to FTLEs and their ridges (locally maximizing curves). We find that in general, the MS and FTLE fields differ and the the FTLE-ridges do not signal MS-ridges.

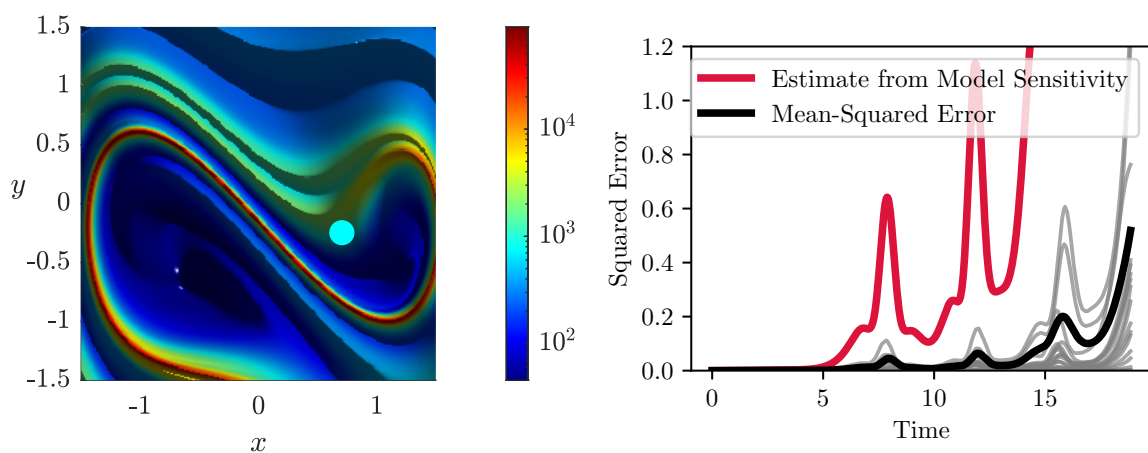


Figure 1: Left panel: MS field of the Duffing oscillator in the two-dimensional phase space, computed over the time interval  $[0, 2\pi]$ . Right panel: Comparison between the leading-order mean-squared prediction error (red curve) and the mean-squared error (black curve) calculated from a Monte Carlo simulation with model errors of magnitude 0.01. The initial condition of the model trajectory is marked in the left panel with a blue circle.

## References

- [1] D. Ruelle, *Nonlinearity* **22**, 855 (2009).
- [2] Q. Wang, R. Hu, and P. Blonigan, *Journal of Computational Physics* **267**, 210 (2014).
- [3] B. Kaszás, G. Haller, Universal upper estimate for prediction errors under moderate model uncertainty, arXiv:2007.07330 (preprint, 2020).