Resonances of the van der Pol Equation with Parametric Damping

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Abstract. Vertical-axis wind-turbine blades undergo large cyclic loading due to the varying fluid flow magnitude and direction relative to the blades, as well as cyclic variations in the damping. Structures in a flow can also have self-excited oscillation. These direct, parametric, and self-excited excitations motivate the consideration of a forced van der Pol equation with cyclic damping. The initial focus of this work entails an analysis of resonances in the parametrically damped van der Pol equation by using the method of multiple scales.

Introduction

Our previous study on vertical-axis wind-turbine blades [1] has shown the existence of a periodic damping in the equation of motion. Allowing for aeroelastic, self excitation is simplified by incorporating quadratic damping, as in a van der Pol equation. The resulting equation of motion in a single-degree-of-freedom transverse deflection, x, is given as $\ddot{x} + \left(F_1 + F_2 x^2\right)\dot{x} + F_3 \dot{x}^3 + \left(F_4 + F_5 x^2\right) x = F_0(t)$, subject to the periodic external force $F_0(t)$ and the time-varying coefficients of similar frequency ω as $F_j = f_{0j} + f_{1j}\cos(\omega t + \phi_j),\ j = 0, 1, \cdots, 5$. In particular, if only $F_0(t) = 0$ and only F_1 is cyclic, the following van der Pol equation with parametric damping

$$\ddot{x} + \epsilon(c_0 + c_1 \cos\omega t + x^2)\dot{x} + {\omega_n}^2 x = 0. \tag{1}$$

The variables c_0 and c_1 are the mean and amplitude of the parametric damping, respectively. The excitation frequency is ω and the natural frequency is ω_n.

Analysis and Results

We seek the approximate solution to Eqn. (1) by using the method of multiple scale [2]. We carry out the analysis up to the first order by considering the two time scales, $T_0 = t$ and $T_1 = \epsilon t$, and therefore, expand the displacement as $x(T_0, T_1) \approx x_0(T_0, T_1) + \epsilon x_1(T_0, T_1)$. The resulting equations are then

$$\epsilon^0 : {D_0}^2 x_0 + {w_n}^2 x_0 = F_0(t) \tag{2}$$

$$\epsilon^1 : {D_0}^2 x_1 + {w_n}^2 x_1 = -2D_0 D_1 x_0 - (c_0 + c_1 \cos\omega T_0 + {x_0}^2)(D_0 x_0) \tag{3}$$

where $D_i = \frac{\partial}{\partial T_i}$. We obtain the solvability conditions by eliminating the secular terms in the right hand side of Eqn. (3) to bound x_1. The relationship between the excitation and the natural frequencies specifies different cases of resonance: the non-resonant case when there is no relationship between ω and ω_n, primary resonance when $\omega \approx \omega_n$, super-harmonic and sub-harmonic resonances when $\omega \approx \omega_n/m$ and $\omega \approx m\omega_n$ ($m \in \mathbb{N}$), respectively.

The objective of this work is to study the primary and secondary resonances and investigate the stability of solution, x, in the space of parameters $\{c_0, c_1, \omega, \omega_n\}$. For example for the van der Pol equation with parametric damping, Eqn. (1), the steady-state amplitude of the response versus the detuning parameter for different values of c_0, the mean damping, is shown in Fig. 1.

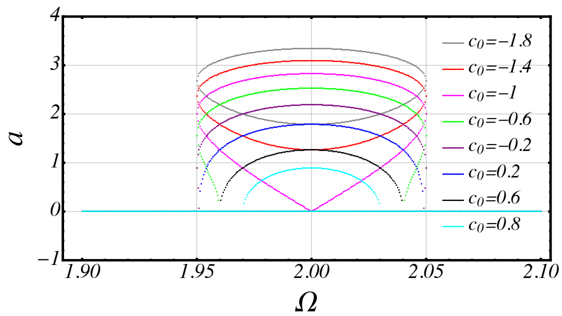

Figure 1: Subharmonic resonance response curve for parametrically damped van der Pol equation.

We also aim to perform similar analysis for a general nonlinear equation, where we add a nonlinear stiffness term and direct excitation.

References

[1] F. Afzali and O. Kapucu and B. F. Feeny, Proceedings of the ASME 2016 International Design Engineering Technical Conferences, August 21-24, Charlotte, North Carolina, paper number IDETC2016-60374, “Vibrational analysis of vertical-axis wind-turbine blades”.

[2] Rand, R. H., 2012. “Lecture notes on nonlinear vibrations, https://ecommons.cornell.edu/handle/1813/28989”.