q-deformations of logistic family

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Abstract. We consider the logistic family f_a and a family of homeomorphisms ϕ_q. The q-deformed system is given by the composition map $f_a \circ \phi_q$. We give an alternative approach to study the dynamics of the q-deformed system with special emphasis on the so-called Parrondo's paradox finding parameter values a for which f_a is simple while $f_a \circ \phi_q$ is dynamically complicated. We explore the dynamics when we apply several q-deformations.

Introduction

By a discrete system we mean a pair (X, f_n), where X is a (compact) metric space and $f_n : X \to X$ is a sequence of continuous maps. Given $x \in X$, its orbit under f is given by the solution of the difference equation $x_{n+1} = f_n(x_n)$, $x_0 = x$. When $f_n = f$ is constant, the pair (X, f) is a classical discrete dynamical system. When the sequence f_n is periodic, we have a discrete periodic system. Along this paper we assume that X is a compact subinterval of the real line. Roughly speaking a q-deformation allow us to modify or introduce a deformation in a discrete dynamical system by using a parameter q such that when q converges to 1 the original discrete dynamical system is obtained. This modified version of the original system can help to study the complexity of the system in the limit and how dynamics changes taking into account the new parameter. Although different functions have been analyzed using q-deformations, see [5] for Hénon map, logistic map and its q-deformations focus a great interest [1, 4].

Results and discussion

The main idea of this work is that we can simplify the analysis of a q-deformed map if we see the deformation as a periodic system of period two. For that, we consider the logistic family $f_a(x) = ax(1-x)$, $a \in [1, 4]$ and a family of homeomorphisms ϕ_q. Following [4], we consider the q-deformation of a real number given by the homeomorphism $\phi_q(x) = \frac{x}{1+(1-q)(1-x)}$, for any $x \in \mathbb{R}$ and $q \in (-\infty, 2)$. Observe that when q tends to 1 then $\phi_q(x)$ tends to x and $\phi_q(0) = 0$ and $\phi_q(1) = 1$, which implies that it is a strictly increasing map for $q < 2$. The q-deformed discrete dynamical system is then $f_a \circ \phi_q$. This system can be derived from the periodic system $(\phi_q, f_a, \phi_q, f_a, ...)$, and shifting this periodic system we obtain the one given by the periodic sequence $(f_a, \phi_q, f_a, \phi_q, ...)$, from which we may derive the map $\phi_q \circ f_a$. We see then that both maps $f_a \circ \phi_q$ and $\phi_q \circ f_a$ have the same dynamical properties, but from a practical point of view it is easier to work with $\phi_q \circ f_a$ instead of $f_a \circ \phi_q$. Hence, we characterize the parameter region for which the q-deformed system is chaotic and check that the Parrondo's paradox (see Figure 1 and [2]) happens in this system, by finding parameter values of a and q such that f_a has a simple dynamics while the dynamics of $\phi_q \circ f_a$ is complicated. Note that the dynamics of ϕ_q is trivially simple and hence, combining two periodic simple maps we obtain a complicated dynamic behavior. We also explore how the number of times that a q-deformation is applied has influence in the Parrondo's paradox. In other words, we analyze the behavior of the periodic system $(\phi_q, ..., \phi_q, f_a, ...)$. Part of this work has been published in [3].

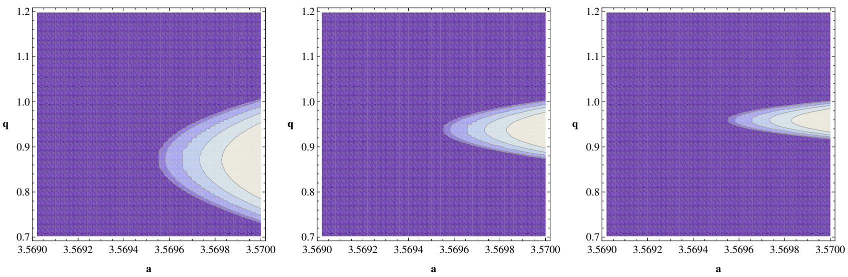

Figure 1: Contour plots of the topological entropy for one (left), two (center) and three (right) maps ϕ_q. We see that q-deformed systems can have positive topological entropy, and hence, a complicated dynamical behavior from dynamically simple single maps. We also show that the parameter region with positive entropy (light color) decreases when we apply several maps ϕ_q.

References

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