Is it always right to judge the dissipativity of a system by divergence?

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Abstract. Starting from two simple examples(a planar linear saddle point system with zero divergence and a planar nonlinear system with a limit cycle), this paper points out that divergence is not a good definition of system dissipation. Therefore, it is necessary to introduce a new criterion, dissipative power, which reveals a very interesting, important and apparently new feature in dynamical systems: To classify dynamics into dissipative or conservative according to the change of "energy function" or "Hamiltonian", not according to the change of phase space volume. Firstly, based on the knowledge of classical mechanics and a novel decomposition of the dynamical system by Ao, a new criterion, dissipative power, is derived. Secondly, we combine with the two examples, analyse and compare these two criteria, find that the dissipative power works when divergence exits contradiction, and obtain the relationship between the dissipative power and Lyapunov function. Finally, the relationship provides a reasonable way to explain why some researchers think that Lyapunov function does not coexist with the limit cycle. Those results may provide a deeper understanding of the dissipation of dynamical systems.

Introduction

Although a lot of achievements have be reached by divergence criterion, it must also be mentioned that, for such important concept, the problem in not-so good definitions have been observed by some researchers. Such as, Chen [1] pointed out that for a system judged as dissipative by divergence, the $div f < 0$ does not necessarily hold at every point in the phase space. More specifically, the planar linear saddle point system with zero divergence, in fact, can be judged not only as a conservative system, but also as a dissipative system. For example, a conservative system defined by the Liouville's Theorem[2] can be dissipative according to the theorem by Borrelli and Coleman[3] that the conservative system has no attractors and repellers (repeller corresponds to an unstable equilibrium point or a saddle point from Sachdev [4]). What's more, Thompson and Stewart[5] observed that the Van der Pol type systems are dissipative, of which, however, the phase space might have regimes of positive divergence. In this paper, we will combine with two simple examples, base on a novel decomposition of the dynamical system of Ao[6, 7] introduced a new one: dissipative power.

Results and discussion

The main results and discussion: (i) For the example of planar linear saddle point system with zero divergence, by dissipative power, it can be judged not only as conservative, but as dissipative. However, the divergence criterion reflects only one situation. Further more, we derive out all the results of the planar linear systems corresponding to the four types of Jordan's normal form, and obtain that these two criteria are consistent except for case $div f = 0$. (ii) For the example of planar nonlinear system with a limit cycle $x_1^2 + x_2^2 = 1$, we obtain its dissipative power $H_P(x) = (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)^2 \geq 0$, and it implies that this system is dissipative and conservative on limit cycle, which will have no contradiction with the theorem by Borrelli and Coleman[3] and can not encounter the problem of Thompson and Stewart[5]. What's more, the meaning of dissipation in an infinitely repeated motion of the limit cycle cannot be explained by $div f = -2$. However, the dissipative power criterion do it: On the limit cycle, the dissipative power $H_p \equiv 0$ indicates that the system is conservative. So the trajectory can move infinitely on it with no dissipation. In addition, from a physical point of view, on limit cycle the potential function is $\phi(x) \equiv -\frac{1}{4}$, which shows that the limit cycle is an isopotential cycle, and the work by electric field force is zero. Therefore, electrons can move infinitely on the isopotential line in the electromagnetic field. (iii) The decrease of Lyapunov function along a trajectory is equal to the dissipative power: $\frac{d\phi}{dt} = -H_p$. The physical meaning is obvious: the decrease of "energy function ϕ" along the system trajectory means dissipation, and no change implies there is no dissipation. (iv) By the relationship between the dissipative power criterion and Lyapunov function in (iii), we give an analysis to the reason why some researchers think that Lyapunov function cannot coexist with the limit cycle.

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