Estimating generic canard explosions via efficient symbolic computation

Bo-Wei Qin[1], Kwok-Wai Chung[2], Antonio Algaba[3] and Alejandro J. Rodríguez-Luis[4]

[1]*School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R. China*

[2]*Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, P.R. China*

[3]*Departamento de Ciencias Integradas, Centro de Estudios Avanzados en Física, Matemática y Computación, Universidad de Huelva 21071, Huelva, Spain*

[4]*Departamento de Matemática Aplicada II, E.T.S. Ingenieros, Universidad de Sevilla, 41092 Sevilla, Spain*

Abstract. In the analysis of canard explosions, asymptotic expansions play a significant role for practical applications. In this work, an efficient analytical algorithm is developed to compute the expansions for a family of generic canard explosions in planar systems. In addition to the applicable formula for the first term, the proposed approach has the capability to find the asymptotic expansions up to any desired order. Moreover, for any system satisfying generic hypothesis, such an algorithm can be readily implemented by the symbolic software *Maple*. To validate the proposed method, the reduced Hodgkin-Huxley model and a templator model are considered as illustrative examples. For both cases, the present algorithm is able to provide exact first non-zero term of the expansion, which improves the existing numerical result in the literature. Furthermore, high-order expansions are also justified by numerical simulations.

Introduction

Canard explosion is referred to a rapid growth of both amplitude and period of the periodic orbit, and is ubiquitous in singularly perturbed systems of diverse disciplines. Several methods have been developed to analyze its existence and asymptotic behavior, such as, non-standard analysis, matched asymptotic expansions and blow-up techniques (see [1] and references therein). On the other hand, finding the critical value for the explosion is also a significant task. Though one can use numerical tools to do so, it is time consuming if high accuracy is needed because the explosion happens in an extremely small interval, and it can only be determined manually. Therefore, an effective high-order approximation is of great significance for applications. There are two main approaches in the literature, namely, the classical iterative method [2, 3] and the nonlinear time transformation (NTT) method [4, 5]. Although it has been proved that the two methods can be applied to find high-order estimations, they still have shortcomings. If the system is relatively complicated, it may not be easy to find the exact value (even for the first term) using the classical iterative method. For instance, only the numeric value is provided in [3], and a quadratic fitting function is used to find the first term in [2] which is not straightforward. The NTT method is proved to be an effective approach for several systems. Nevertheless, its full generality has not been confirmed yet.

Results and discussion

Here, we consider the canard explosion in the system $\epsilon\dot{x} = f(x, y, \mu, \epsilon)$, $\dot{y} = g(x, y, \mu, \epsilon)$, with $0 < \epsilon \ll 1$, and develop a symbolic computational approach to find asymptotic expansions up to any desired order. A simple formula is also afforded for the first term. The approach is adapted from the NTT method without using trigonometric functions. In such a way, it is easier to be implemented in the symbolic software *Maple* and is more efficient. More importantly, it can be applied to any system satisfying generic hypothesis which suggests its potential and extensive applications. With the present approach, the asymptotic expansions of critical values for the templator model and the reduced Hodgkin-Huxley model are greatly improved (see Fig. 1).

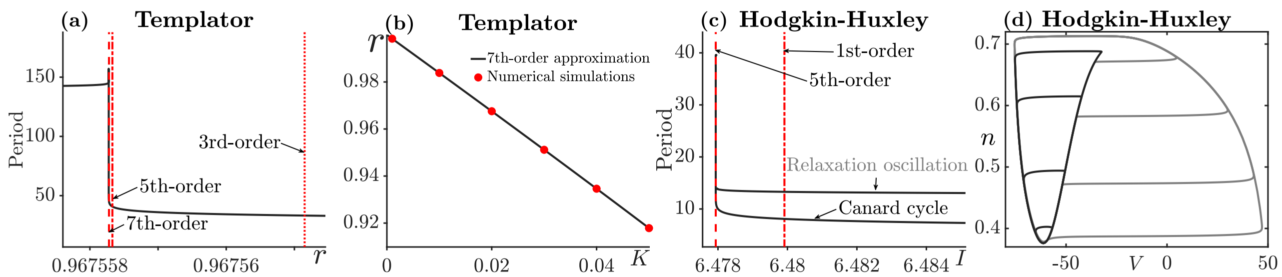

Figure 1: Analytical and numerical results for the canard explosions. (a) and (b): comparisons in the templator model, (c): comparisons in the reduced HH model, (d): the evolution of the orbits from canard cycle to relaxation oscillation in the reduced HH model.

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