Expansion of evolution matrix and Lyapunov exponents with respect to parameters

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Abstract. For a regular dynamical system governed by ordinary differential equations formulas for expansion of the Lyapunov exponents (LEs) with respect to small change in parameters are derived and expressed through the derivatives of the LEs with respect to parameters and approximation of evolution matrix with averaging method. We consider discrete QR algorithm for computing all of the LEs and obtain approximate change of the LEs if there is a small variation of system parameters. These novel LE approximations could simplify the analysis of system's behavior in the space of parameters.

Introduction

Consider linearization, $\dot{p}(t) = \mathbf{J}(t)\, p(t)$, of a nonlinear system about its solution, where $p(t) \in \mathbb{R}^n$ is the vector of state variable perturbations and $\mathbf{J}(t)$ is piecewise continuous, integrable *Jacobian matrix* of the original nonlinear system. Solution of the matrix differential equation $\dot{\mathbf{P}}(t) = \mathbf{J}(t) \cdot \mathbf{P}(t)$ with the initial identity matrix $\mathbf{P}(0) = \mathbf{I}$ yields *evolution matrix* $\mathbf{P}(t)$. LEs are defined as logarithms of eigenvalues of matrix $\Lambda = \lim_{t\to\infty} \left(\mathbf{P}'(t) \cdot \mathbf{P}(t)\right)^{\frac{1}{2t}}$, where $\mathbf{P}'$ is the transposed matrix $\mathbf{P}$ and we assume that the finite limit exists (system is regular). QR factorization of $\mathbf{P}(t) = \mathbf{Q}(t) \cdot \bar{\mathbf{R}}(t)$ yields orthogonal matrix $\mathbf{Q}(t)$, upper triangular matrix $\bar{\mathbf{R}}(t)$ with positive diagonal elements, and expression for i-th LE as $\lambda^{(i)} = \lim_{t\to\infty} \frac{1}{t} \log |\bar{\mathbf{R}}^{(i,i)}(t)|$, where $i = 1, \ldots, n$ and ${}^{(i,i)}$ denotes i-th diagonal element. When the largest LE is positive column-vectors in matrix $\mathbf{P}(t)$ undergo exponential growth and become numerically linearly dependent as $t \to \infty$. That is why QR factorization is done periodically, as in [1], by the Gram–Schmidt algorithm: $\mathbf{P}(t) = \mathbf{X}(t) \cdot \mathbf{Q}'(k-1) \cdot \mathbf{P}(k-1)$,

$$\dot{\mathbf{X}}(t) = \mathbf{J}(t) \cdot \mathbf{X}(t), \quad \lim_{t\downarrow k-1} \mathbf{X}(t) = \mathbf{Q}(k-1), \quad t \in (k-1, k], \quad k = 1, 2, 3, \ldots, \quad \mathbf{Q}(0) = \mathbf{I}, \tag{1}$$

where $\mathbf{Q}(k)$ is the result of Gram–Schmidt orthonormalization of column vectors in matrix $\mathbf{X}(k)$ and it is the next initial value (right-sided limit as $t \downarrow k$) of matrix function $\mathbf{X}$. We have $\mathbf{X}(k) = \mathbf{Q}(k) \cdot \mathbf{R}(k)$ and $\mathbf{P}(k) = \mathbf{X}(k) \cdot \mathbf{Q}'(k-1) \cdot \mathbf{P}(k-1) = \mathbf{Q}(k) \cdot \mathbf{R}(k) \cdot \mathbf{Q}'(k-1) \cdot \mathbf{P}(k-1) = \mathbf{Q}(k) \cdot \bar{\mathbf{R}}(k)$, where $\bar{\mathbf{R}}(k) = \prod_{j=k}^{1} \mathbf{R}(j)$. Because $\mathbf{R}(k)$ is upper triangular, we have LE as $\lambda^{(i)} = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \log \mathbf{R}^{(i,i)}(k)$.

Results and discussion

We solve problem (1) approximately by the averaging method [3] in [4] assuming that the Jacobian matrix $\mathbf{J}(t)$ can be expanded into the series $\mathbf{J}(t) = \mathbf{J}_0(t) + \mathbf{J}_1(t) + \ldots$, with the lower index denoting the order of smallness, so we know solution $\mathbf{X}_0(t)$ of the matrix initial value problem $\dot{\mathbf{X}}_0(t) = \mathbf{J}_0(t) \cdot \mathbf{X}_0(t)$ for $t \in (k-1, k]$, where $\lim_{t\downarrow k-1} \mathbf{X}_0(t) = \mathbf{Q}_0(k-1)$, $\mathbf{Q}_0(0) = \mathbf{I}$, and $k = 1, 2, 3, \ldots$. For $t \in [0, 1]$ we have the expansion of the evolution matrix till the first order term $\mathbf{P}(t) = \mathbf{X}(t) \approx \mathbf{X}_0(t) \cdot \left(\mathbf{I} + \int_0^t \mathbf{X}_0^{-1}(\tau) \cdot \mathbf{J}_1(\tau) \cdot \mathbf{X}_0(\tau)\, \mathrm{d}\tau\right)$, that could also be valid for $t \in [0, \infty)$ under additional conditions not assumed here, see, e.g. [3]. First two terms of LE expansion $\lambda^{(i)} = \lambda_0^{(i)} + \lambda_1^{(i)} + \ldots$ are $\lambda_0^{(i)} = \lim_{K\to\infty} \frac{1}{2K} \sum_{k=1}^{K} \log(\mu_i(k))$ and $\lambda_1^{(i)} = \lim_{K\to\infty} \frac{1}{2K} \sum_{k=1}^{K} \frac{\mathbf{v}_i'(k) \cdot \mathbf{M}_1(k) \cdot \mathbf{v}_i(k)}{\mu_i(k)\left(\mathbf{v}_i'(k) \cdot \mathbf{v}_i(k)\right)}$, where $\mathbf{v}_i(k)$ and $\mu_i(k) = |\mathbf{R}_0^{(i,i)}(k)|^2$ are the i-th eigenvector and the i-th eigenvalue of matrix $\mathbf{M}_0(k) := \mathbf{X}_0'(k) \cdot \mathbf{X}_0(k) = \mathbf{R}_0'(k) \cdot \mathbf{R}_0(k)$, so that $\mathbf{M}_0(k) \cdot \mathbf{v}_i(k) = \mathbf{v}_i(k)\, \mu_i(k)$. Expression for $\lambda_1^{(i)}$ is derived with the use the derivative of LEs in [2], where $\mathbf{M}_1(k) = \mathbf{A}_1'(k) \cdot \mathbf{M}_0(k) + \mathbf{M}_0(k) \cdot \mathbf{A}_1(k)$ and $\mathbf{A}_1(k) = \int_{k-1}^{k} \mathbf{X}_0^{-1}(t) \cdot \mathbf{J}_1(t) \cdot \mathbf{X}_0(t)\, \mathrm{d}t$. The novelty here is in applying the averaging scheme that automatically removes in $\mathbf{M}_1(k)$ higher order terms which may appear in Taylor expansions, [2, 5]. This work was supported by Russian Science Foundation, grant 19-11-00223.

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