The Wave Front Tracking Method and Delta Shocks

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Abstract. The subject of this paper is numerical verification of delta shock waves interaction for a pressureless gas dynamics system using the approximated Wave Front Tracking algorithm. The system is perturbed with a small pressure term becoming strictly hyperbolic. The new bounds on the strength of elementary waves in the Wave Front Tracking algorithm is obtained as well as a global solution for initial data with arbitrary large total variation, provided that the perturbing parameter is small enough.

Introduction

There are many papers where one can find examples of systems that have no classical solution of Riemann problem (i.e. Cauchy initial value problem). Most of these solutions are delta waves ([3]) or singular shock waves ([4]). Bressan [2] generalized Wave Front Tracking (WFT) method for genuine nonlinear systems without restriction in the number of equations of conservation laws. The basic algorithm will be the one presented by Asakura [1]. The original problem is modified with the introduction of perturbation. The main purpose of this paper is to extend the idea from [3] to a more general initial data than Riemann ones using the ideas from [1] and [4]. Our first result in that direction, is an improvement in the estimate of the strength of waves in the WFT method from [1]. Replacing κ^2 from [1] with $\varepsilon/(1+2\,\varepsilon)$, precise calculation shows that the strength of wave is now bounded by $\sqrt{\varepsilon}$ instead of ε. That permits us to take the initial data with large total variation and let $\varepsilon \to 0$ after solving (1) as it was done in [3]

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \varepsilon\rho^\gamma/\gamma) &= 0, \end{aligned} \tag{1}$$

where $\rho \geq 0$ is the density, u is the velocity with $\varepsilon > 0$ being a small parameter and $\gamma > 1$ is a parameter depending on ε.

Results and discussion

The main idea is the following: for $\varepsilon > 0$ small enough, we set Riemann problems

$$(\rho, u)\,|_{t=0} = \begin{cases} U_0 = (\rho_0, u_0), & x < a_1, \\ U_1 = (\rho_1, u_1), & a_1 < x < a_2, \\ U_2 = (\rho_2, u_2), & x > a_2, \end{cases} \tag{2}$$

for (1) such that each solution consists of shock waves of the first and second family (this approximates the interaction of two delta shock waves). A *delta shock* is an SDW associated with a δ distribution with minor components having finite limits as $\varepsilon \to 0$. Next, we apply the WFT method for each ε, and finally let $\varepsilon \to 0$. The solution is obtained when there are no further interactions. It has been shown that result of two delta shocks interaction is a delta shock with constant speed in a special case. From the numerical results, as $\varepsilon \to 0$, the density of the intermediate state in obtained solutions increases as $1/\varepsilon$, while the velocity becomes closer to the predicted value. Finally, we have the main result:

Theorem 0.1 *For arbitrary initial data with the bounded total variation, there exists ε_0 such that for each $\varepsilon < \varepsilon_0$ WFT scheme remains stable and provides a global in time solution for the one-dimensional Euler small pressure gas dynamics system. For each $t > 0$, the sum of strengths of shock waves does not exceed* $\min\{1/2, C^*/4\}/(C_*\sqrt{\varepsilon})$.

Assume that (ρ_0, u_0) can be connected to (ρ_2, u_2) by a single delta shock (so-called simple SDW). The result of the interaction is a single SDW with a constant speed c_δ. Take (1,2) where $a_1 = 0$, $a_2 = 2$, $(\rho_0, u_0) = (1, 1)$, $(\rho_1, u_1) = (1.2, 0.8)$ and $u_2 = 0.7$. Two SDWs will interact at point $(X, T) = (12.635, 13.809)$. There exists a single simple SDW as a solution to the interaction problem for $\rho_2 = 1.14286$. After the interaction, the correct speed of the resulting wave equals $c_\delta = 0.84499$. At Figure 1, the solutions to the Riemann problem (1,2) for different ε at $t = 5735$ are given.

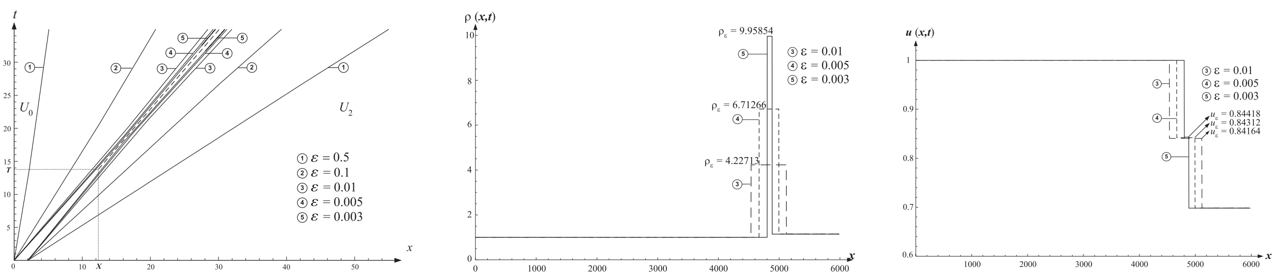

Figure 1: Phase $x - t$ plane, solution $\rho(x,t)$ and $u(x,t)$ to (1,2) for different ε at $t = 5735$.

References

[1] F. Asakura, *Wave-front tracking method for the equations of isentropic gas dynamics*, Quart. Appl. Math. 63 (2005), no. 1, 20–33.

[2] A. Bressan, *Global solutions of systems of conservation laws by wave-front tracking*, J.Math.Anal.Appl. 170, 414-432, 1992.

[3] G.Q. Chen and H. Liu, *Formation of delta-shocks and vacuum states in the vanishing pressure limit of solutions to the isentropic Euler equations*, SIAM J. Math. Anal. 34 (2003), no. 4, 925–938.

[4] M. Nedeljkov, *Delta and singular delta locus for one dimensional systems of conservation laws*, Math. Methods Appl. Sci. 27 (2004), no. 8, 931–955.