Externally Excited Purely Nonlinear Oscillators:
Exact Steady-State Solutions, Their Approximations and Further Benefits

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Abstract. This work is concerned with an analytical technique that one can use to find an exact steady-state response of externally excited oscillators whose restoring force contains a single power-form nonlinear term. Multiple benefits of knowing such solutions exist, including the derivation of known approximate solutions for certain values of the power.

Introduction

Purely nonlinear oscillators under consideration are governed by the equation:

$$\ddot{x} + 2\delta\dot{x} + \mathrm{sgn}(x)|x|^{\alpha} = F(\Omega t), \tag{1}$$

where α is any positive real number, δ is constant, and F stands for periodic external excitation of frequency Ω. Many techniques have been developed for obtaining their free response ($F = 0$) (see, for example, [1] or [2] and the references cited therein). Externally excited have been treated for the case of harmonic external excitation $F = F_0\cos(\Omega t)$, by assuming a generating solution corresponding to δ=$F_0 = 0$ in various forms - harmonic or elliptic [2], and proceeding by averaging or perturbation-like techniques as for linear oscillators. This work, however, focuses on obtaining the exact solution of Eq. (1) for a certain form of periodic excitation.

Results and discussion

The solution for free response of purely nonlinear oscillators, Eq. (1) with δ=$F = 0$ can be expressed as a closed-form solution in terms of the Ateb ca function: $x = A\,\mathrm{ca}(\alpha, 1, \omega_{\mathrm{ca}} t)$, the frequency of which is $\omega_{\mathrm{ca}} = |A|^{\frac{(\alpha-1)}{2}}\sqrt{(\alpha+1)/2}$, where A is the amplitude. In order to obtain the solution for the undamped forced case first, the excitation is assumed in the form related both to the form of the restoring force with the amplitude F_0, as well as to the form of this exact solution for the free response, but with a new frequency ω_{r} introduced, yielding the frequency-amplitude equation:

$$-\frac{2}{\alpha+1}\omega_{\mathrm{r}}^2 A + \mathrm{sgn}(A)|A|^{\alpha} = F_0. \tag{2}$$

This simplifies to the known solutions for $\alpha = 1$ and $\alpha = 3$ [2]. Going back to the Ateb ca form and developing it into series for the latter case, one can obtain the known approximation found by a perturbation approach [3]:

$$x \approx AC_1\cos\left(\frac{\pi}{2K}\omega_{\mathrm{r}} t\right) + AC_3\cos\left(3\frac{\pi}{2K}\omega_{\mathrm{r}} t\right) + AC_5\cos\left(5\frac{\pi}{2K}\omega_{\mathrm{r}} t\right), \tag{3}$$

where C_1, C_3, C_5 are certain constants, while $K \equiv K(1/2)$ is the complete elliptic integral of the first kind.

Besides this benefit to cover the existing approximate solution, this technique also yields the exact solutions for the steady-state response and the frequency-amplitude equation for any α, and also leads to other benefits [2], such as: obtaining the exact solution for the damped case, and deriving isochronous (amplitude-independent) solutions, as illustrated in Figure 1.

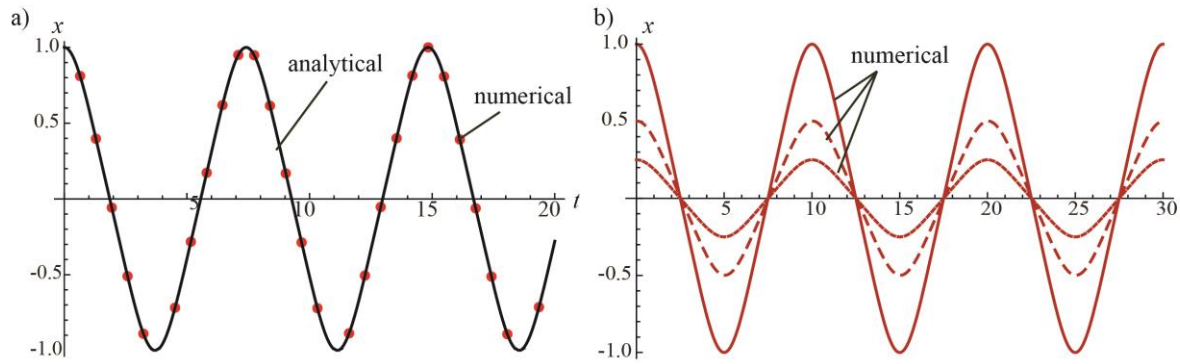

Figure 1: Numerical and exact analytical solutions of Eq. (1) for the cubic case: a) damped; b) isochronous.

References

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