Construction of soliton solutions of the matrix modified Korteweg-de Vries equation

Sandra Carillo[*] and Cornelia Schiebold [**]

[*] *Dipartimento di "Scienze di Base e Applicate per l'Ingegneria",*
LA SAPIENZA - *Università di Roma, Rome, Italy*
&
I.N.F.N. - Sezione Roma1, Gr. IV - M.M.N.L.P., Rome, Italy
[**] *Department of Science Education and Mathematics Mid Sweden University, Sundsvall, Sweden*

Abstract. Nonlinear evolution equations known also as *non-commutative soliton equations* are considered. In particular, the matrix modified Korteweg-deVries (mKdV) equation is studied. Based on Bäcklund transformations which allow to reveal algebraic properties as well as construct solutions of nonlinear evolution equations, a method to construct explicit solutions is presented. The produced solutions can be considered as *soliton* solutions; they exhibit the typical behaviour of solitons. Indeed, they are shown to represent a generalisation of the corresponding scalar solutions. Finally, perspectives concerning the construction of soliton solutions admitted by further matrix equations are given.

Introduction

Matrix equations are an example of non-commutative equations with a notable applicative meaning. Indeed, quantum mechanical systems are governed by matrix equations. On the other hand, matrix equations can be regarded as a special case of operator equations, when finite dimensional operators are considered.
The present note is concerned about the matrix mKdV equation

$$V_t \;=\; V_{xxx} + 3\{V^2, V_x\} \;\;, \;\; \text{where} \;\; \{A, B\} := AB + BA, \;\; \forall A, B. \tag{1}$$

The unknown V is represented by a $n \times n$ matrix, further developing results in [4]. Notably, we obtain solutions of the matrix mKdV are linked, via Bäcklund transformations to solutions of the matrix KdV equation which are within the solution class of multisoliton solutions obtained, via IST (inverse scattering transform [1]) method by Goncharenko [12] in the case of the matrix KdV equation. Conversely, our viewpoint is to apply Bäcklund transformations [13] to construct solutions as well as to reveal symmetry properties enjoyed by third order matrix equations. This approach generalises the study in [11] where scalar third order nonlinear evolution equations are considered. Analogies and differences between commutative and non-commutative cases are analysed to [2, 7]. The present investigation is based on previous results [3, 5, 6, 9, 10], where a key role is played by Bäcklund transformations. The aim is to briefly illustrate a method to construct *soliton* solutions. Furthermore, we report on recent progress in [8] and discuss various examples.

References

[1] F. Calogero and A. Degasperis, Spectral Transform and Solitons I, *Studies in Mathematics and its Application*, Vol. 13, North Holland, Amsterdam, 1980.

[2] S. Carillo, *KdV-type equations linked via Bäcklund transformations: remarks and perspectives*, Applied Numerical Mathematics, **141** (2019) pp. 81-90.

[3] S. Carillo, M. Lo Schiavo, E. Porten, C. Schiebold, *A novel noncommutative KdV-type equation, its recursion operator, and solitons*, J. Math. Phys., **59**, (4), (2018), 14 pp.

[4] S. Carillo, M. Lo Schiavo, C. Schiebold, *Matrix solitons solutions of the modified Korteweg-de Vries equation*, NODYCON 2019 Springer Proceedings, B. Balachandran, J. Ma, W. Lacarbonara, G.Quaranta, J. Machado, G. Stepan, Ed.s, (2020), 75-83.

[5] S. Carillo, M. Lo Schiavo, C. Schiebold, Bäcklund Transformations and Non Abelian Nonlinear Evolution Equations: a novel Bäcklund chart, *SIGMA*, 12 (2016), 087, 17 pages, doi:10.3842/SIGMA.2016.087.

[6] S. Carillo, M. Lo Schiavo, C. Schiebold, Recursion Operators admitted by non-Abelian Burgers equations: Some Remarks, Mathematics and Computers in Simulation, **147C**, (2018) pp. 40–51.

[7] S. Carillo, M. Lo Schiavo, C. Schiebold, *Abelian versus non-Abelian Bäcklund Charts: some remarks*, Evolution Equations and Control Theory, **8** (1), (2019), 43-55.

[8] S. Carillo, M. Lo Schiavo, C. Schiebold, *N-soliton matrix mKdV solutions: a step towards their classification* (2020), preprint.

[9] S. Carillo, C. Schiebold, Non-commutative KdV and mKdV hierarchies via recursion methods, *J. Math. Phys.* **50** (2009), 073510.

[10] S. Carillo, C. Schiebold, Matrix Korteweg-de Vries and modified Korteweg-de Vries hierarchies: Non-commutative soliton solutions, *J. Math. Phys.* **52** (2011), 053507.

[11] B. Fuchssteiner, S. Carillo, Soliton structure versus singularity analysis: Third order completely integrable nonlinear equations in 1+1-dimensions, *Phys. A* **154** (1989), 467-510.

[12] V.M. Goncharenko, *Multisoliton solutions of the matrix KdV equation*,Theor. Math. Phys. **126**, (2001).

[13] C. Rogers, W.K. Schief, *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory*, Cambridge University Press, Cambridge, 2002.