# A New Technique for the Approximate Analysis of Quasi-Periodic Systems

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**Abstract**. The paper presents a new technique for the approximate analysis of ordinary differential equations with quasi-periodic coefficients (so-called quasi-periodic systems). The technique utilizes Lyapunov-Perron transformation to reduce the linear part of a quasi-periodic system into the time-invariant form. The resulting dynamical equation is then analyzed using time-invariant methods. The usefulness of the technique is shown by examining two distinct quasi-periodic systems.

## Introduction

Lyapunov-Perron (L-P) transformations [1] reduce the linear part of a quasi-periodic system into the time-invariant form. Such transformations can be useful for the study of quasi-periodic systems as many existing techniques applicable to time-invariant systems can be used for such problems. For instance, bifurcation studies require nonlinear equations of the perturbed dynamics. If the linear part of the equations can be made time-invariant via L-P transformations, these nonlinear equations can be simplified using local nonlinear techniques such as time-dependent normal form theory and center manifold reduction. Furthermore, controllers can be designed using time-invariant methods. Unfortunately, there are no methods available to compute L-P transformations. Therefore, an attempt has been made in this paper to compute L-P transformations for general quasi-periodic systems. Recently, Sharma and Sinha [2] proposed an approach to compute approximate state transition matrices of general quasi-periodic systems. This paper utilizes their approach to construct approximate L-P transformations and their inverses. The usefulness of such transformations is demonstrated by investigating two different forms of the following quasi-periodic system.

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & 1 \\ -\left(a + b\cos\omega_1 t + b\cos\omega_2 t\right) & -d \end{bmatrix}}^{A(t)} x + G(t) + W(x,t) \quad ; \quad \omega_1 = 1 \text{ and } \omega_1 = \left(1+\sqrt{5}\right)/2 \tag{1}$$

where $a, b$ and $d$ are the system parameters, $t$ is the time, $G(t)$ is a force vector and $W(x,t)$ is a nonlinear vector containing homogenous monomials in $x$ with quasi-periodic coefficients. Following the approach proposed in ref. [2], Eq. (1) is replaced by the following periodic system.

$$\dot{x} = \bar{A}(t)x + G(t) + W(x,t) \tag{2}$$

where $A(t)$ is approximated by $\bar{A}(t)$ such that $\bar{\omega}_1 = 1.0031058$ and $\bar{\omega}_2 = 1.6161149$. $\bar{A}(t)$ is periodic with the principal period, $T_a = 112.747$. Readers are advised to see ref. [2] for details. Application of the transformation, $x = Q(t)z$ reduces Eq. (2) to

$$\dot{z} = Rz + Q^{-1}(t)G(t) + Q^{-1}(t)W(Qz,t) \tag{3}$$

where $Q(t)$ is an approximate L-P transformation and $R$ is a constant matrix. Eq. (3) is amenable to the direct application of time-invariant methods.

## Results and Discussion

First, a forced quasi-periodic Hill equation (Eq. (1) with $W(\bullet) = 0$) is investigated and approximate analytical solutions are generated when the transient part of the solution is asymptotically or critically stable. It is found that the closeness of the analytical solutions to the exact solutions computed numerically depends on the principal period of the periodic system. A nonlinear quasi-periodic system (Eq. (1) with $G(\bullet) = 0$) is also analyzed to show the usefulness of approximate L-P transformations. Once the linear part of the periodic system has been reduced to the time-invariant form, the resulting system i.e., Eq. (3) can be analyzed using the time-dependent normal form theory. Equation (1) with cubic nonlinearity is examined to demonstrate the effectiveness. The closed-form solution generated is found to be in good agreement with the exact solution.

## References

[1] Bogoljubov N.N., Mitropoliskii Ju. A., Samoilenko A.M. (1976) Methods of Accelerated Convergence in Nonlinear Mechanics. Hindustan Publishing Corporation, Delhi.

[2] Sharma A., Sinha S.C. (2018) An Approximate Analysis of Quasi-Periodic Systems via Floquét Theory. *J Comput Nonlin Dyn* **13(2)**: 021008.