A multi-time Hamiltonian approach to the derivation of a high-order nonlinear Schrödinger equation for the envelope of slowly modulated wave trains

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Abstract. We propose a multi-time Hamiltonian approach to the derivation of a high-order nonlinear Schrödinger equation for the complex amplitude of the envelope of the slowly modulated wave trains whose evolution is governed by the nonlinear Klein–Gordon equation. To this end, we derive and solve a system of multi-time Hamilton equations for the Fourier amplitudes and polymomenta of the wave field's envelope. Such a multi-time Hamiltonian representation of the nonlinear wave equation is of significance in various problems of classical and quantum physics.

Introduction

The nonlinear Schrödinger equation (NLSE) describes the slowly varying envelope of a rapidly oscillating nonlinear wave train, which is strongly dispersive, nearly monochromatic, and weakly nonlinear [1]. The classical cubic NLSE includes two basic terms: the second-order dispersion and cubic nonlinearity. High-order NLSE models include high-order dispersive, nonlinear, and nonlinear dispersive terms. Such high-order NLSEs are often derived by proper balancing of high-order terms with respect to multiple timescales. In particular, such a multi-scale approach was used to derive a sixth-order NLSE for the complex amplitude of the envelope of the slowly modulated wave trains whose evolution is governed by the nonlinear Klein–Gordon (nKG) equation with arbitrary polynomial nonlinearity [2, 3]. To this end, the multiple-scale expansions in terms of wave envelope's small amplitude were performed either straightforwardly in the equations of motion [2] or in the associated Lagrangian [3], with the both approaches producing exactly the same results. The purpose of this work is to use yet another approach and derive a high-order NLSE by employing multiple-scale expansions in the Hamiltonian associated with the original nKG equation.

Results and discussion

We start from the nKG equation and expand the unknown function of the wave field in a Fourier series with slowly modulated coefficients and rapidly oscillating phase. We use the method of multiple scales to separate slow and fast motions. Next we derive a system of multi-time Hamilton equations for the complex Fourier amplitudes of the wave field function and the associated polymomenta. Such systems of multi-time Hamilton equations are often referred to as the De Donder–Weyl equations [4]. In our multi-time representation, these equations are written down in each order of smallness with respect to the complex amplitude of the first-harmonic envelope. The Hamilton equations in the lowest (second) order of smallness yield the linear dispersion relation. The third-order approximation yields the linear wave equation for the complex amplitude of the first-harmonic envelope and a relation for the group speed. The fourth-order approximation yields the classical cubic NLSE. High-order NLSEs are obtained as Hamilton's equations in each of the subsequent orders of smallness. The coefficients of the resulting high-order NLSE coincide with those derived earlier based on the multi-scale expansions of the nKG equation [2] or its associated Lagrangian [3]. As compared to the Lagrangian approach, the use of multi-time Hamilton equations does not require averaging over the rapid phase.

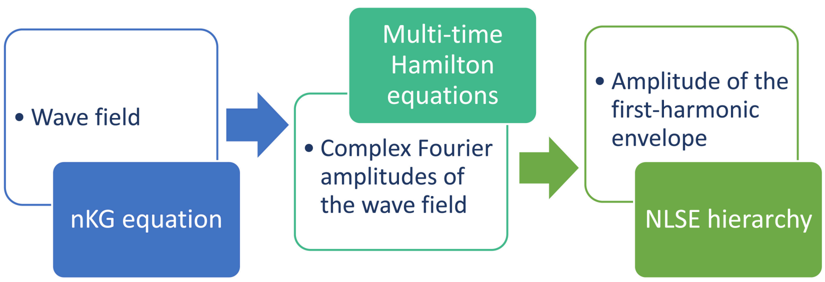

Figure 1: Graphical abstract.

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