# A hybrid averaging and harmonic balance method for asymmetric systems

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**Abstract**. We introduce a new technique that provides relatively simple approximations for the free and forced vibration response of weakly nonlinear systems, including those with asymmetric restoring forces. For free vibration, it captures the correct amplitude-frequency dependence, including cases of non-monotonicity. The method can also be used to determine the steady-state response of damped, harmonically driven vibrations, including stability results. The method is a blend of a first order perturbation calculation with higher order harmonic balance (HB), carried out by amplitude expansions. The HB aspect of the method captures information about higher harmonic overtones and the constant (DC) offset. General results are derived for an asymmetric system with up to quintic nonlinear stiffness terms. The results are validated using simulations. This approach will be useful for analyzing a variety of system models with polynomial nonlinearities.

## Introduction

We consider weakly nonlinear vibration models with asymmetric restoring forces of the form

$$\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 = F\cos(\omega t) \tag{1}$$

which arise in numerous applications. The structure of the steady state forced vibration response curves of such systems depends on the amplitude-frequency backbone curve obtained for free undamped ($F = 0, \Gamma = 0$) vibrations. These backbones can exhibit non-monotonic amplitude-frequency relationships. Accurate descriptions of such a backbone curve and of the forced response typically require tedious calculations, e.g., higher order perturbation methods (cf. [1]) or the use of action-angle coordinates (cf. [2]). In the present work we derive a method that is a combination of higher order HB and first order averaging to obtain accurate results for these systems with significantly less effort.

## Results and Discussion

The solution process proceeds by assuming that the response is dominated by the fundamental harmonic with slowly varying amplitude $a$ and phase $\phi$. It is also assumed that the amplitudes and phases of the higher harmonics (HH) and the constant offset (DC) adiabatically track $(a, \phi)$ and have transients on a timescale that is neglected in the analysis. The HB method is applied using amplitude expansions, which provides closed form results for the DC and HH terms. These are then used in a standard first order averaging formulation to obtain the slow flow equations

$$\dot{a} = -\Gamma a - \frac{F}{2\omega_0}\sin\phi, \quad \dot{\phi} = \omega_0 - \omega + \frac{3\gamma_{\text{eff}}}{8\omega_0}a^2 + \frac{5\sigma_{\text{eff}}}{16\omega_0}a^4 - \frac{F}{2a\omega_0}\cos\phi,$$

where

$$\gamma_{\text{eff}} = \alpha_3 - \frac{10}{9}\left(\frac{\alpha_2}{\omega_0}\right)^2 \quad , \quad \sigma_{\text{eff}} = \alpha_5 - \frac{11}{12}\left(\frac{\alpha_2^2}{\omega_0^3}\right)^2 + \frac{53\alpha_3}{20}\left(\frac{\alpha_2}{\omega_0^2}\right)^2 - \frac{14\alpha_2\alpha_4}{5\omega_0^2} + \frac{3}{80}\left(\frac{\alpha_3}{\omega_0}\right)^2.$$

It is important to note that these equations do not correctly capture the transient dynamics of $(a, \phi)$, but accurately predict their steady state values and their stability.

The backbone curves are described by the $\dot{\phi}$ equation with $F = 0$, $\omega = \omega_0$. Sample response curves obtained from this method for a model with only quadratic and cubic nonlinearities are shown in the Figure (details will be provided in the presentation). The stable (unstable) branches of the response curves are denoted by solid (dashed) curves. The symbols ► and ◄ indicate the fundamental harmonic amplitude of the steady-state obtained from simulations from sweep-up and sweep-down, respectively, obtained by time integration of Eq.(1) and computing the fundamental harmonic of the steady-state response. Note that the non-monotonicity feature of this system is not captured by standard second order perturbation methods, which provide only the $\gamma_{\text{eff}}$ term [3]. The example demonstrates the ability of the method to capture non-monotonic behavior using a first order perturbation method. This method will be useful for describing the frequency response of asymmetric systems in terms of physical system parameters.

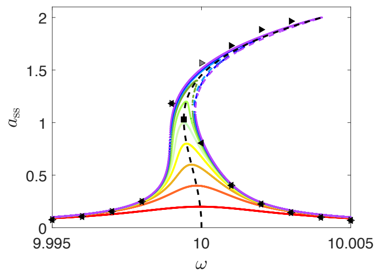


Figure 1: Sample response curves.

## References

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