Nonlinear interactions of widely spaced modes

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Abstract. Nonlinear interactions of modes with vastly different eigenfrequencies (VDE) are ubiquitous and occur in various fields of engineering and physics. We show that the complex dynamics of these interactions can be distilled into a single generic form, namely, the Stuart-Landau oscillator. We use this model to study frequency combs that arise from injection locking and frequency pulling of a driven low-frequency (LF) mode that interacts with a driven blue-detuned high-frequency (HF) mode. Our analysis shows that the frequency combs are tunable around both the high and low carrier frequencies. The novelty of our analysis lies in the minimalistic conceptual view that it offers, including an analogy between these complex interactions and a simple mechanical system model and a connection with the motion of an overdamped particle in a tilted washboard potential. The results are applicable to a wide class of systems and suggest means of generating tunable frequency combs.

Introduction

We consider a model of the form

$$\ddot{q}_0 + 2\Gamma_0\dot{q}_0 + \omega_0^2 q_0 + \alpha q_1^2 = F_0\cos(\omega_{F_0}t), \quad \ddot{q}_1 + 2\Gamma_1\dot{q}_1 + \omega_1^2 q_1 + 2\alpha q_0 q_1 = F_1\cos(\omega_{F_1}t), \tag{1}$$

where $\omega_0 \ll \omega_1$. Eq. (1) can be used to model the dynamics of widley spaced modes in mechanical macro-structures [1]; cavity optomechanics and plasmomechanics, where the interactions are between HF optical modes and the LF mechanical modes; interactions between HF nano- and LF micro-mechanical modes; certain classes of aeroelastic instabilities, such as stall flutter and transverse galloping, where these interactions are between HF vortex modes of the turbulent wake (the so-called Kármán vortex street) and the LF modes of the elastic structure; and many other systems.

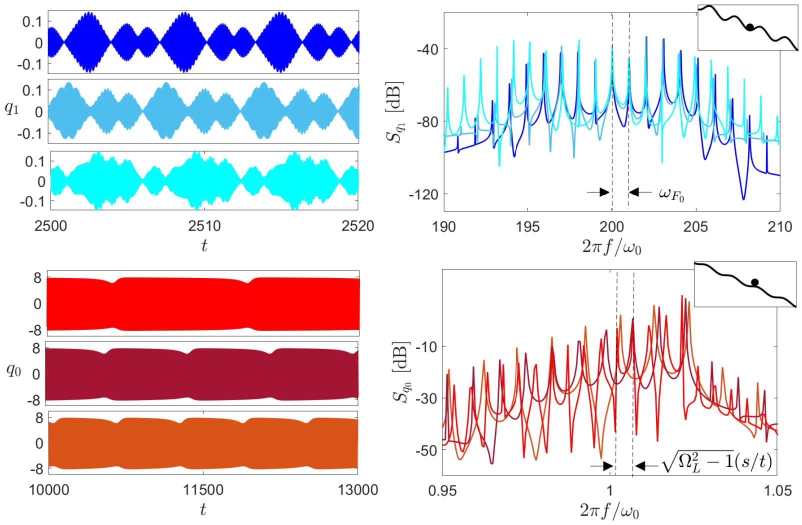

Figure 1: Injection locking (top panel) and pulling (bottom panel) of the LF mode.

Results and Discussion

The HF modes typically decay faster than the LF modes. Thus, we assume that $\Gamma_1 \gg \Gamma_0$, and therefore, q_1 adiabatically tracks q_0 when $t \gg \Gamma_1^{-1}$. In this case, the complex-amplitude equation of the LF mode obeys the Stuart–Landau oscillator $\dot{A}_0 = (\sigma - l|A_0|^2)A_0 - F_0/(4\omega_{F_0})$. Using polar notation for the complex amplitude of the LF mode $A_0 = -a_0^{[\varphi_0+\arg(\ell)]}/2$, we find that under certain conditions (which will be specified in the presentation), the phase dynamics are governed by the Adler equation $d\varphi_0/ds = \Omega_L - \sin\varphi_0$, where $s = [F_0|\ell|/(4\omega_{F_0}\sqrt{\Re\{\sigma\}\Re\{\ell\}})]t$ is a non-dimensional time and $\Omega_L = 4\omega_{F_0}\Im\{\ell^*\sigma\}\sqrt{\Re\{\sigma\}/\Re\{\ell\}}/(F_0|\ell|)$ is the non-dimensional one-sided frequency-locking range (i.e., the overall locking range of the LF mode is $\pm\Omega_L$ around $\omega_0 t/s$). The phase dynamics of φ_0 determines the conditions for injection locking and pulling (Fig. 1). A constant phase $\dot{\varphi}_0 = 0$ (inset of the right top panel) of the injection-locked LF mode generates periodic modulations in the HF mode (top left panel), which corresponds to a frequency comb around ω_1 with a spacing of ω_{F_0} in the power spectral density of q_1 (top right panel). An unlocked phase $\dot{\varphi}_0 \neq 0$ (inset of the right bottom panel) of the injection-pulled LF mode is periodically modulated in a highly non-uniform rate (bottom left panel), which corresponds to a frequency comb around ω_0 with a spacing of $\sqrt{\Omega_L^2 - 1}(s/t)$ in the power spectral density of q_0 (bottom right panel).

References

[1] A. H. Nayfeh and D. T. Mook, (1995) *Energy Transfer from High-Frequency to Low-Frequency Modes in Structures*, J. Vib. Acoust., 117, 186-195