Predicting limit cycle of modified Rayleigh differential equation

V. Shrikanth*, Amar K. Gaonkar* and Pramod Kumar Verma*
*Department of Mechanical, Materials and Aerospace Engineering,
Indian Institute of Technology Dharwad, India, ORCID 0000-0001-9820-173X

Abstract. The Rayleigh differential equation is used to model the oscillations of a clarinet reed. Surprisingly, the same equation with a small modification can be used to model Coulomb friction. This modification can accurately model weakly nonlinear friction systems. The Rayleigh equation having cubic nonlinearity in velocity gives rise to an unstable limit cycle. In this work, an analytical expression is proposed to predict the said limit cycle, which holds good even for strong nonlinearity. The analytical expression is a third order polynomial in α, which is the damping constant.

Introduction

The general equation of motion of a dry friction oscillator in its nondimensionalized form is $z''+\beta\mathrm{sgn}(z')+z=\tilde{f}(\tau)$. By the method of harmonic balancing [1] and assuming the system to have a periodic motion [2], the signum function can be approximated to the third harmonic to express the equation of motion as

$$z''+\alpha\left(z'-(2/3)(z')^3\right)+z=\Gamma\cos\Omega\tau \tag{1}$$

where α is damping constant, Γ is the forcing amplitude and Ω is the frequency, all nondimensional parameters. The ranges are $0<\alpha<1$, $0<\Gamma<1$ and $\Omega\in\mathbb{R}^+$. Eq.(1) is the modified Rayleigh equation.

Results and Discussion

The state space description of the Eq.(1) under autonomous condition is given by

$$\dot{u}=v \quad \text{and} \quad \dot{v}=-u-\alpha v\left(1-(2/3)v^2\right) \tag{2}$$

where $u=z$ and $v=z'$. The derivatives in Eq.(2) are with respect to τ. There is only one equilibrium/fixed point at $(u^*,v^*)\equiv(0,0)$ found at the intersection of the two nullclines $\dot{u}=0$ and $\dot{v}=0$. Existence of limit

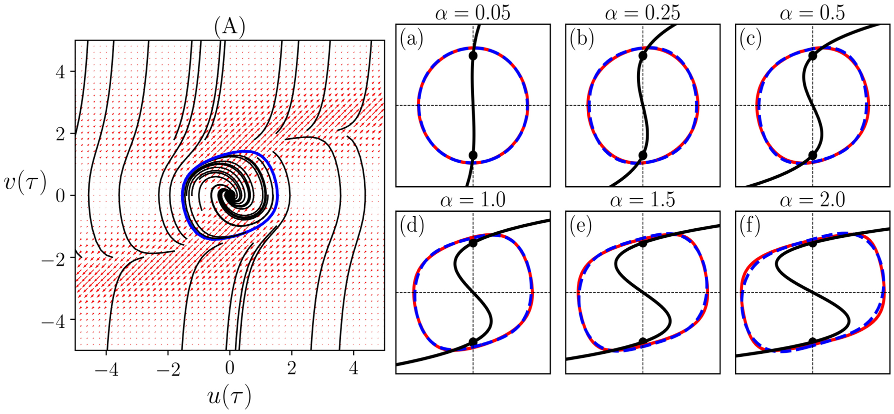

Figure 1: (A) Phase trajectories and flow field for the Eq.(1) without any external forcing. The autonomous system has an unstable limit cycle represented by the blue coloured line. (a)–(f) Comparison of numerical and analytical limit cycles and nullcline $\dot{v}=0$ for varying values of α

cycle [3], approximate limit cycle [4] and Hopf bifurcation [5], from a state of no motion at all to a state of self-oscillations at a fixed amplitude. An analytical equation to express the limit cycle(s) can be derived to encompass the cases presented in Fig. 1(a)–(f). The equation should reduce to a circle when $\alpha\to 0$ as shown in Fig. 1(a), while the same equation should be able to represent the limit cycle shown in Fig. 1(f). The equation for the limit cycle is as follows:

$$u^2\left(\alpha u^2+1\right)+\gamma v^2-\delta uv=\chi \tag{3}$$

where $\gamma\equiv\gamma(\alpha)$, $\delta\equiv\delta(\alpha)$ and $\chi\equiv\chi(\alpha)$. The ordinates of the limit cycle(s) is always greater than $\tilde{v}$ and approaches $\tilde{v}$ when $\alpha\gg 1$. In order to accommodate the rotation of the limit cycle which is evident in Fig. 1(d)–(f), the coordinates u and v can be replaced by $\bar{u}=u\cos\varphi+v\sin\varphi$ and $\bar{v}=v\cos\varphi-u\sin\varphi$ with $\varphi\equiv\varphi(\alpha)$. The variation of the parameters in Eq.(3) as a function of the damping term α can be expressed as $\gamma=1.871\alpha^3-1.014\alpha^2+2.573\alpha+0.914$. Similarly, the variation of δ with α is $\delta=0.462\alpha^3-0.469\alpha^2+0.966\alpha-0.052$. The variation of φ against α is $\varphi=-0.0065\alpha^2+0.0374\alpha+0.1301$. Dependence of χ on α is $\chi=2.685\alpha^3-0.958\alpha^2+3.891\alpha+1.946$.

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