Exact potentials in multivariate Langevin equations

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Abstract. Systems governed by a multivariate Langevin equation featuring an exact potential exhibit straightforward dynamics but are often difficult to recognize because, after a general coordinate change, the gradient flow becomes obscured by the Jacobian matrix of the mapping. In this work, a detailed analysis of the transformation properties of Langevin equations under general nonlinear mappings is presented. We show how to identify systems with exact potentials by understanding their differential-geometric properties. To demonstrate the power of our method, we use it to derive exact potentials for broadly studied models of nonlinear deterministic and stochastic oscillations. In selected examples, we visualize the potentials to illustrate how our method enables new analytical descriptions of the classic phenomena of beating, synchronization and symmetry breaking. Our results imply a broad class of exactly solvable stochastic models which can be self-consistently defined from given deterministic gradient systems.

Introduction

In stochastic dynamical systems featuring exact potentials driven by white noise, the evolution of a n-dimensional set of variables $x = (x_1, \ldots, x_n)^T \colon \mathbb{R} \to \mathbb{R}^n$ over time $t \in \mathbb{R}$ is governed by following the Langevin equation:

$$\dot{x} = -\nabla \mathcal{V}(x,t) + \Xi. \tag{1}$$

In Eq. (1), $\dot{()}$ is the time derivative, ∇ is the gradient operator, $\mathcal{V} \colon \mathbb{R}^n \to \mathbb{R}$ is the potential, $\mathcal{F}_i = -\nabla_i \mathcal{V}(x,t)$ is the ith component of the restoring force $\mathcal{F}$ and the vector $\Xi = (\xi_1, \ldots, \xi_n)^T \colon \mathbb{R} \to \mathbb{R}^n$ contains uncorrelated white noise sources ξ_i, $i = 1, \ldots, n$ of equal intensity Γ. The individual entries ξ_i of Ξ are assumed to be delta-correlated: $\langle \xi_i \xi_{i,\tau} \rangle = \Gamma \delta(\tau)$, where $\langle \cdot \rangle$ is the expected value operator, $(\cdot)_{,\tau}$ denotes a positive time shift by τ and δ is the Dirac delta function. In particular, we are concerned with identifying the presence of an underlying exact potential in general noise-driven systems taking the form

$$\dot{x} = \mathcal{F}(x,t) + \mathcal{B}(x)\Xi, \tag{2}$$

where $\mathcal{F}$ is a vector- and $\mathcal{B}$ a n-by-n tensor field. With the knowledge of $\mathcal{F}$, one can easily deduce if an exact potential $\mathcal{V}$ exists for x by checking the following necessary and sufficient conditions: $\nabla_i \mathcal{F}_j = \nabla_j \mathcal{F}_i$ for all i and $j \neq i$. However, if these conditions are not fulfilled, this does not preclude the existence of an exact potential governing the original variables that were transformed into x via a certain nonlinear mapping. We therefore argue that, instead of applying the above conditions, Eq. (2) should be compared to a Langevin equation with potential *after* a coordinate change defined by an arbitrary nonlinear mapping $x = f(y)$, see Fig. 1. Below, we give a brief summary of our main results.

Results

Assuming purely additive noise in the equations governing the underlying potential system which transforms objectively under local rotations and reflections, the resulting transformed Langevin equation with potential reads, after redefining $y \to x$,

$$\dot{x} = -g^{-1}(x)\nabla\widetilde{\mathcal{V}}(x,t) + h^{-1}(x)\Xi, \tag{3}$$

where the Jacobian of f, $J(x) = \nabla f(x)$, was assumed to be nonsingular (invertible) with polar decomposition $J = Qh$, $Q = Q^{-T}$ is orthogonal, $g = h^T h$ is the positive definite metric tensor, $(\cdot)^T$ is the transpose, h is a positive definite matrix and $\widetilde{\mathcal{V}}(x,t) = \mathcal{V}\big(f(x), t\big)$ is the transformed potential.

In this work, we derive necessary and sufficient conditions for the existence of an exact potential in a noise-driven system given by Eq. (2). Our results imply a self-consistent way of modeling noise in given deterministic gradient flows $\dot{x} = -\nabla\mathcal{V}(x,t)$, and the resulting models are exactly solvable if the potential is stationary.

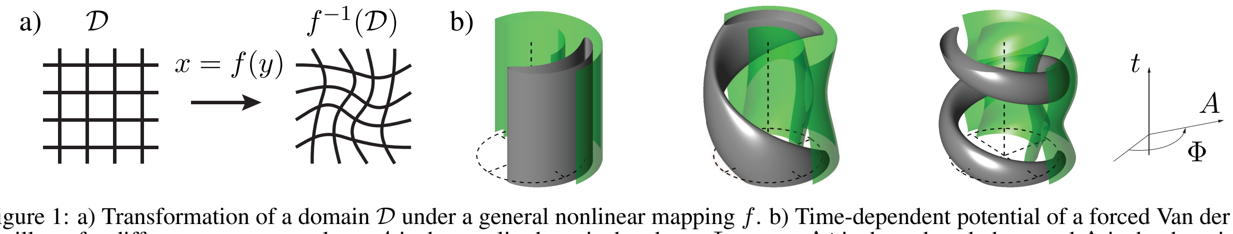

Figure 1: a) Transformation of a domain $\mathcal{D}$ under a general nonlinear mapping f. b) Time-dependent potential of a forced Van der Pol oscillator for different parameter values. A is the amplitude, φ is the phase, $\Phi = \varphi - \Delta t$ is the reduced phase and Δ is the detuning.

References

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