Non-autonomous inverse Jacobi multipliers and periodic orbits of planar vector fields

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Abstract. We analyze the role that non-autonomous (and not necessarily periodic) inverse Jacobi multipliers have in the problem of the nonexistence, existence and localization as well as the hyperbolic nature of periodic orbits of planar vector fields. This work generalizes and extends previous results already appearing in the literature focusing in the autonomous or periodic case.

Introduction

We consider planar differential systems

$$\dot{x} \;=\; P(x,y), \;\; \dot{x} \;=\; Q(x,y), \tag{1}$$

where P and Q are C^1 functions on the open set $U \subseteq \mathbb{R}^2$ and the dot denotes, as usual, derivatives with respect to the independent time variable t. We associate to system (1) the vector field $\bar{\mathcal{X}} = P(x,y)\partial_x + Q(x,y)\partial_y$ and state the precise definition of inverse Jacobi multiplier of system (1).

DEFINITION. A function $V : \mathbb{R} \times U \to \mathbb{R}$ is said to be an inverse Jacobi multiplier of system (1) if V is of class $C^1(\mathbb{R} \times U)$, it is not locally null and it satisfies the following linear first order partial differential equation:

$$\mathcal{X}(V) = V\, \mathrm{div}(\mathcal{X}), \tag{2}$$

where $\mathcal{X} = \partial_t + \bar{\mathcal{X}}$ and the divergence of $\mathcal{X}$ is $\mathrm{div}(\mathcal{X}) = \mathrm{div}(\bar{\mathcal{X}}) = \partial_x P + \partial_y Q$.

We use the name autonomous inverse Jacobi multiplier V in case that $\partial_t V \equiv 0$ and periodic inverse Jacobi multiplier when there is $T > 0$ such that $V(T,x,y) = V(0,x,y)$ for all $(x,y) \in U$.

Results and discussion

Let $\Phi(t;(x,y)) \subset U$ be the flow associated to $\bar{\mathcal{X}}$ with $\Phi(0;(x,y)) = (x,y) \in U$. Given a T-periodic orbit Γ of $\bar{\mathcal{X}}$, we consider $\Sigma \subset \mathbb{R}^2$, a transversal section to the flow of $\bar{\mathcal{X}}$ with one endpoint $p_0 \in \Gamma$. We parameterize $\Sigma = \{(\bar{x}(s), \bar{y}(s)) \in U : 0 \leq s \ll 1\}$ such that $p_0 = (\bar{x}(0), \bar{y}(0))$. Let $\mathcal{P} : \Sigma :\to \Sigma$ be the *Poincaré map* associated to Γ, that is, $\mathcal{P}(s) = \Phi(\tau(s); (\bar{x}(s), \bar{y}(s)))$ where τ is the first time return function which is the unique function such that $\tau(0) = T$, the period of Γ, and $\Phi(\tau(s); (\bar{x}(s), \bar{y}(s))) \in \Sigma$. Clearly $\mathcal{P}(0) = 0$ and the orbit Γ is hyperbolic if $\mathcal{P}'(0) \neq 1$ where the prime indicates derivative with respect to s.

Our main result is the forthcoming Theorem.

THEOREM. *Assume that there exists a T-periodic orbit $\Gamma \subset U \subset \mathbb{R}^2$ of the C^1-vector field $\bar{\mathcal{X}}$ defined in U and let $\mathcal{P} : \Sigma \to \Sigma$ be its Poincaré map associated to a transversal section $\Sigma \subset U$ parameterized by $0 \leq s \ll 1$ where $s = 0$ indicates the point $\Sigma \cap \Gamma$. Let $V(t,x,y)$ be an inverse Jacobi multiplier of $\bar{\mathcal{X}}$ defined in $\mathbb{R} \times U$. Then*

$$V(T,x,y) = V(0,x,y)\, \mathcal{P}'(0) \tag{3}$$

for any point $(x,y) \in \Gamma$.

Some related references are given below.

References

[1] Berrone L.R. and Giacomini H (2003), Inverse Jacobi multipliers, *Rend. Circ. Mat. Palermo* (2) **52**, 77–130.

[2] Buică A. and García I.A. (2015), Inverse Jacobi multipliers and first integrals for nonautonomous differential systems, *Z. Angew. Math. Phys.* **66**, 573-585.