The integral of the cofactor as a characterization of centers

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Abstract. In this work we deal with analytic families of real planar vector fields $\mathcal{X}_\lambda$ having a monodromic singularity at the origin for any $\lambda \in \Lambda \subset \mathbb{R}^p$. There naturally appears the so-called center-focus problem which consists in describing the partition of Λ induced by the centers and the foci at the origin. We give a characterization of the centers (degenerated or not) in terms of a specific integral of the cofactor associated to a real invariant analytic curve which always exists.

Introduction

We consider families of real analytic planar differential systems

$$\dot{x} = P(x, y; \lambda), \;\; \dot{y} = Q(x, y; \lambda), \tag{1}$$

with parameters $\lambda \in \mathbb{R}^p$ together with its associate family of vector fields $\mathcal{X}_\lambda = P(x, y; \lambda)\partial_x + Q(x, y; \lambda)\partial_y$. Throughout the work we restrict the family to the parameter space $\Lambda \subset \mathbb{R}^p$ so that the origin $(x, y) = (0, 0)$ is made sure to be a *monodromic singularity* of the whole family (1). That means that $P(0, 0; \lambda) = Q(0, 0; \lambda) = 0$ and the local orbits of $\mathcal{X}_\lambda$ turn around the origin for any $\lambda \in \Lambda$. In this monodromic scenario, I'lyashenko [3] and Écalle [1] show independently that the origin only can be either a center or a focus since $\mathcal{X}_\lambda$ is analytic. We recall that a *center* possesses a punctured neighborhood (period annulus) foliated by periodic orbits of $\mathcal{X}_\lambda$ while in a neighborhood of the *focus* the orbits spiral around it.

Main results

We prove the following results.

THEOREM 1. *Let $\mathcal{X}$ be real analytic planar vector field with coprime components and having a monodromic singular point at the origin. Then there exists a real analytic invariant curve $F(x, y) = 0$ of $\mathcal{X}$ with $F(0, 0) = 0$ and F having an isolated zero in $\mathbb{R}^2$ at the origin.*

Let $(p, q) \in W(\mathbf{N}(\mathcal{X}))$ be two weights associated to the Newton diagram $\mathbf{N}(\mathcal{X})$ of $\mathcal{X}$ and perform the *weighted polar* blow-up $(x, y) \mapsto (\rho, \varphi)$ given by $(x, y) = \phi(\varphi, \rho) = (\rho^p \cos\varphi, \rho^q \sin\varphi)$ transforming (1) into the polar vector field $\dot{\rho} = R(\varphi, \rho)$, $\dot{\varphi} = \Theta(\varphi, \rho)$ and consider the differential equation

$$\frac{d\rho}{d\varphi} = \mathcal{F}(\varphi, \rho) := \frac{R(\varphi, \rho)}{\Theta(\varphi, \rho)} \tag{2}$$

well defined in $C\backslash\Theta^{-1}(0)$ being the cylinder $C = \left\{(\theta, \rho) \in \mathbb{S}^1 \times \mathbb{R} : 0 \le \rho \ll 1\right\}$ with $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$.
Let $F(x, y) = 0$ be a real invariant analytic curve of $\mathcal{X}$ with cofactor K, that is, $\mathcal{X}(F) = KF$. In weighted polar coordinates this equation is transformed into $\hat{\mathcal{X}}(\hat{F}) = \hat{K}\hat{F}$ where $\hat{\mathcal{X}} = \partial_\varphi + \mathcal{F}(\varphi, \rho)\partial_\rho$, $\hat{F} = F \circ \phi$ and $\hat{K}$ is the cofactor of the curve $\hat{F} = 0$.
Let $\rho(\varphi; \rho_0)$ be the solution of the Cauchy problem (2) with initial condition $\rho(0; \rho_0) = \rho_0 > 0$ and small, and $\gamma_{\rho_0} = \{(\varphi, \rho(\varphi; \rho_0)) : 0 \le \varphi \le 2\pi\} \subset C$ an arc of orbit of (2). We define $\int_{\gamma_{\rho_0}} \hat{K} = PV \int_0^{2\pi} \hat{K}(\varphi, \rho(\varphi; \rho_0))\, d\varphi$, where PV stands for the Cauchy principal value.

THEOREM 2. *Let $\mathcal{X}$ be a family of analytic planar vector fields having a monodromic singular point at the origin and K the cofactor associated to an analytic invariant curve. Then $\int_{\gamma_{\rho_0}} \hat{K}$ exists and moreover the origin is a center if and only if*

$$\int_{\hat{\gamma}_{\rho_0}} \hat{K} \equiv 0 \tag{3}$$

for any initial condition $\rho_0 > 0$ sufficiently small.

The talk is extracted from a preprint [2] still not submitted to any journal.

References

[1] J. Écalle, Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac. Actualités Mathématiques. Hermann, Paris, 1992.
[2] García I.A and Giné J. (2022) Stability of degenerate monodromic singularities by its complex separatrices. Preprint. University of Lleida.
[3] Yu.S. Il'yashenko, Finiteness theorems for limit cycles. Translated from the Russian by H. H. McFaden. Translations of Mathematical Monographs, **94**. American Mathematical Society, Providence, RI, 1991.