A perturbation theory for the shape of central force orbits

Ritapriya Pradhan*, Tanushree Bhattacharya* and Jayanta K. Bhattacharjee *

*School of Physical Sciences, Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700032, INDIA

Abstract. The two body central force orbit can be solved exactly only for the gravitational and simple harmonic oscillator potentials. When one discusses nonlinear oscillators, the trajectory in space-time can be found by various kinds of perturbative techniques- one of the most prominent ones being the Lindstedt-Poincare perturbation theory. In this work we show that a Lindstedt–Poincare like perturbation theory can be set up for the shape of a general central force orbit by working round a circular orbit. One also gets an answer for spatial frequency by this process. The effectiveness of our technique is checked against numerical simulations.

Introduction

We consider the dynamics of a particle of mass 'm' moving in a central force field where the force is taken to be of the form $F = -m\lambda r^{-n}$, where n is any number such that bound orbits exist. The distance of the particle from the center of force is 'r' and 'λ' is the interaction strength. The conservation of the angular momentum (magnitude 'l' per unit mass) restricts the particle to a plane. In terms of the polar co-ordinates 'r' and 'θ' , we have ($u = 1/r$)

$$\frac{d^2u}{d\theta^2} + u = \frac{\lambda}{l^2}u^{n-2} \tag{1}$$

Our perturbation theory is set up around the circular orbit characterized by $u_0 = (\frac{\lambda}{l^2})^{\frac{1}{3-n}}$. The energy of the orbit is $E_c = \frac{1}{2}l^2u_0^2\frac{n-3}{n-1}$. Appropriate modifications are necessary for $n = 1$. The deviation u_1 from the circular orbit defines the dimensionless quantity $X = \frac{u_1}{u_0}$. The variable X satisfies the dynamics

$$\frac{d^2X}{d\theta^2} + (3-n)X = \sum_{k=2}^{\infty} {}^{n-2}C_k X^k \tag{2}$$

The energy is expressed in terms of X as

$$\Delta E = E - E_c = \frac{1}{2}l^2u_0^2\left[\left(\frac{dX}{d\theta}\right)^2 + 2X + X^2 - \frac{2}{n-1}\left\{(1+X)^{n-1} - 1\right\}\right] \tag{3}$$

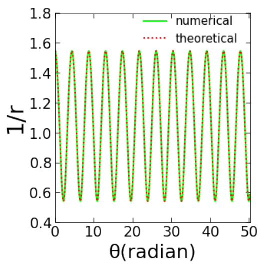
We have thus reduced the orbit equation formally to an anharmonic oscillator equation with coordinate X and timelike variable θ. The order of perturbation theory is determined by how many powers of X is retained. In some ways this is another example of a traditional perturbation theory being used in an unexpected situation [1].

Results and Discussion

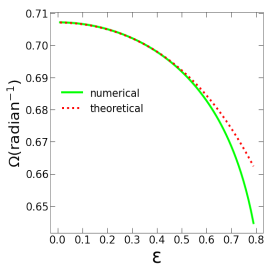
The orbit upto second order in ϵ is (initial conditions suitably chosen)

$$u = \left(\frac{\lambda}{l^2}\right)^{\frac{1}{3-n}}\left[1 - \epsilon^2\frac{n-2}{4} + \epsilon\cos\left(\sqrt{3-n}\,\theta\right) + \epsilon^2\frac{n-2}{12}\cos\left(2\sqrt{3-n}\,\theta\right)\right] \tag{4}$$

One gets a spatial frequency $\Omega = \sqrt{3-n}$ within this order. We get corrections to this as we go to higher order. Note, ϵ is the order of amplitude of X and hence is the perturbation parameter. Our results agree with the exact solutions for $n = 2$ and $n = -1$. The comparison between our perturbation theory result and the numerically obtained trajectory, spatial frequency is shown in Figure 1.



(a)



(b)

Figure 1: (a) Plot of $\frac{1}{r}$ as a function of θ for $n = 1$, $\epsilon = 0.5$, $u_0 = 1$ (b) Plot of Ω as a function of ϵ for $n = 2.5$

References

[1] T Shah, R.Chattopadhyay, K.Vaidya, S.Chakraborty (2015) Conservative Perturbation Theory for Non Conservative Systems. *Phys Rev E* **92** 062927