Mirroring of synchronization in multilayer configuration of Kuramoto oscillators

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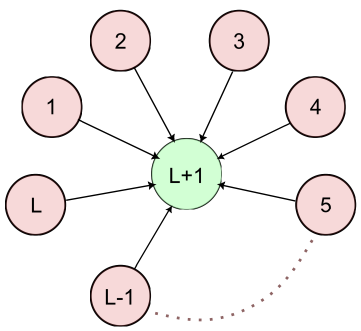
Abstract. In certain dynamical systems, a phenomena is observed where emergent dynamical behavior in one part of the system is seen to propagate into another part that is only weakly coupled to the first. This can be defined as *mirroring*. To model this, we propose a multilayer system of globally coupled Kuramoto oscillators with $(L+1)$ layers, with the first L layers being the master layers, all of which drive the $(L+1)^{\text{th}}$ layer, which is the slave layer. It implies that the dynamics of the $(L+1)^{\text{th}}$ layer is not only determined by it's own properties but also affected by each of the L master layers, which leads to the phenomena of mirroring. The properties of this model is investigated and results for mirroring in the slave layer is derived for different values of L as a function of other system parameters and verified using simulations.

Introduction

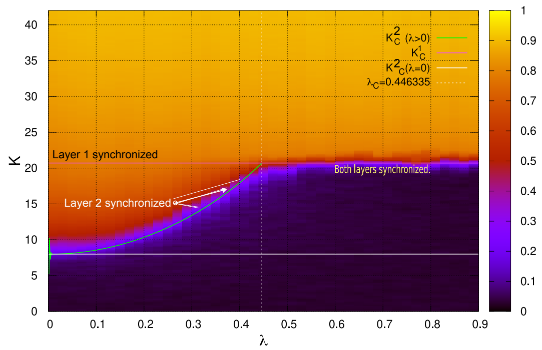
In this work, we investigate a multilayer network of globally coupled Kuramoto oscillators with $(L+1)$ equally populated layers. The layers from 1 to L represent the independent master layers, whereas the $(L+1)^{\text{th}}$ layer represents the slave layer (see Fig.(1a)). The dynamics of the slave layer depends not only on it's own properties but also on the unidirectional interlayer coupling that exists with all the master layers. Here, the focus is on "mirroring", where dynamical phenomena (like synchronization) in the master layers is *mimicked* in the slave layer and we derive the conditions for it's occurrence. The dynamics of each master layer is governed by the globally coupled Kuramoto model [1], whereas the dynamics of the slave layer is described by

$$\dot{\phi}_i^{(L+1)} = \sum_k \lambda^{(k)} \left[\omega_i^{(k)} + \frac{K}{N}\sum_j \sin(\phi_j^{(k)} - \phi_i^{(k)})\right] + \left(1 - \frac{1}{L}\sum_k \lambda^{(k)}\right)\left[\omega_i^{(L+1)} + \frac{K}{N}\sum_j \sin(\phi_j^{(L+1)} - \phi_i^{(L+1)})\right]$$

for $i, j = 1, 2, ..., N$ and $k = 1, 2, ..., L$. Here, $\phi_i^{(l)}$ and $\omega_i^{(l)}$ denotes the phase and natural frequency respectively of the i^{th} oscillator in the l^{th} layer, $\lambda^{(l)}$ quantifies the contribution of the l^{th} master layer to the slave layer, K denotes the global coupling strength and N is the number of oscillators per layer.



(a) Topology (nodes → layers)



(b) Variation of $r^{(2)}$ as a function of (K, λ)

Figure 1

Results and Discussions

Preliminary results are obtained for L=1 where we compute the synchronization order parameter [1] of the slave layer (say, $r^{(2)}$) as a function of both K and λ=$\lambda^{(1)}$ (see Fig.(1b), [2]) for a generalized Cauchy distribution of natural frequencies [3] with different shape and scale factors. A sharp boundary in the color map is observed, denoting the value of K=$K_C^{(2)}$ for which the slave layer transitions to partial synchronization. $K_C^{(2)}$ follows a non-linearly increasing function in the range of $\lambda \in (0, \lambda_C]$, the analytical expression [2] for which is plotted as a green solid line showing fair agreement with numerical simulations. For $\lambda > \lambda_C$, $K_C^{(2)}$ becomes constant at $K_C^{(2)}$=$K_C^{(1)}$, thus indicating mirroring. Generalization of these results for $L>1$ are currently in progress.

References

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