Short-Time Angular Impulse Response of a Rayleigh Beam

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Abstract. The short-time impulse response of slender structures like beams is of both practical and academic interest. The Euler-Bernoulli beam model excited by *angular* impulses exhibits unbounded response. Whereas, the addition of rotary inertia (aka Rayleigh beam) yields bounded response for a similar excitation. The response (central angular impulse on a simply supported Rayleigh beam) correspondingly is given by an anharmonic Fourier series. We use asymptotics to find the first few terms in an expansion of the response, valid for short times. Our work complements earlier work on Euler-Bernoulli beams with *linear* impulses, and provides a satisfactory example of such analysis for a series expansion. Since adding small modal damping does not change the short-time response much, our results have practical utility as well.

Introduction

The short-time responses of infinite plates and beams to linear impulses have been studied earlier [1, 2]. An infinite Euler-Bernoulli beam yields an unbounded response to an angular impulse. Adding rotary inertia, i.e., adopting Rayleigh beam theory, we obtain physically more reasonable results. The governing equation for such a beam excited by angular impulse is

$$\rho A\, u_{tt} + EI\, u_{xxxx} - \rho I\, u_{xxtt} = -M_0\, \delta_x \left(x - \frac{L}{2} \right) \delta(t), \tag{1}$$

where the symbols and notations have their usual meaning. The coefficients on the left hand side can be set to unity by, e.g., choice of units; M_0 can be set to unity owing to linearity and length L is taken to be π for analytical convenience. However, length has no influence on the short-time response since wave speeds in a Rayleigh beam are bounded. Elementary methods then yield the angular response as

$$u_x \left(x = \frac{\pi}{2}, t \right) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{\sqrt{1+4\,k^2}} \sin \left(\frac{4\,k^2}{\sqrt{1+4\,k^2}}\, t \right) = \frac{2}{\pi}\, S(t), \;\text{ say.} \tag{2}$$

The short-time behavior of the solution in Eq. 2 is not obvious. Numerically summed responses, with and without a small amount (1%) of modal damping are shown in Fig. (1a). Also shown in Fig. (1b) is an asymptotic approximation of the short-time undamped response,

$$S(t) \sim \frac{\pi}{4} - \frac{\pi \coth\left(\frac{\pi}{2}\right) t}{4} + \frac{3\,\pi\, t^2}{16} + \left(-\frac{1}{6} + \frac{-5\,\pi\, \sinh(\pi) + \pi^2 + 8\, \cosh(\pi) - 8}{-48 + 48\, \cosh(\pi)} \right) t^3 + \mathcal{O}\left(t^4\right). \tag{3}$$

The above asymptotic series was obtained by splitting the sum into two: the first part consisting of terms

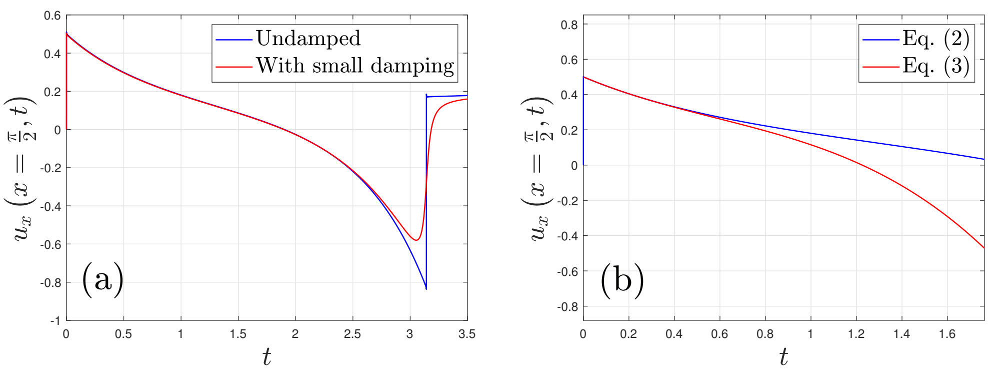

Figure 1: Response of a simply supported beam subjected to angular impulse at its midpoint: (a) Undamped and damped response, (b) Asymptotic approximation.

such that large-k asymptotics begin to hold although $kt \ll 1$, and the second part consisting of the remaining terms, approximated using an integral plus correction terms. Collecting terms of relevant orders yields the above approximation. More terms can in principle be computed. Although the system studied herein is linear, we hope that the asymptotic analysis presented will be of interest to the nonlinear dynamics community. Additionally, the initially discontinuous rotation which then evolves smoothly is of theoretical interest in structural dynamics.

References

[1] Zener, C., The intrinsic inelasticity of large plates. *Physical Review*, 59(8): 669, (1941).

[2] Chatterjee, A., The short-time impulse response of Euler-Bernoulli beams. *Journal of Applied Mechanics, ASME*, 71(2): 208-218, (2004).