Stochastic symplectic Ito-Taylor based integrator for non-linear oscillators on S^2

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Abstract. The application of stochastic non-linear oscillators towards the modelling of complex non-linear phenomena has made them pervasive in science and engineering. For naturally occurring systems and modern applications, existing non-linear oscillators have been extended to evolve on differentiable manifolds. This work presents a symplectic numerical integrator based on Ito-Taylor formulation for the stochastic non-linear oscillators evolving on the configuration manifold S^2, with uncertainty in momentum, which also preserves the geometry of the manifold. The symplectic nature of the scheme is assessed by preserving drift in the Hamiltonian that results from the continuous pumping of energy owing to the stochastic nature of the states.

Introduction

Non-linear systems evolving on manifold unfolds an intriguing research domain with application towards theoretical sciences and applied mathematics, and their dynamics can naturally be described using the Hamiltonian function. Literature [1] treats the distance function in the system potential as a Riemannian distance function, laying the groundwork for the extension of known potential functions to Riemannian manifolds and eventually defining standard non-linear oscillators on the manifold. Although the development of numerical integration schemes for these oscillators exists in the literature, they fail to consider the geometry of the manifold [2], symplecticity of the integrator [3] and stochasticity of the system [2] all together. This motivates the current work to develop a symplectic numerical integration scheme for the non-linear stochastic oscillators evolving on a manifold. The Riemannian distance function is given as $\mathfrak{m}(a,b) = \|\log_a(b)\|_a$, where $\log(\cdot)$ is the manifold logarithm. The present study validates the symplectic scheme through a stochastic Duffing Van-Der-Pol oscillator, evolving on S^2. Kinetic and potential energy for the DVP oscillator on S^2 are defined as $\frac{1}{2}\langle\pi,\pi\rangle$ and $\frac{1}{2}\alpha\mathfrak{m}^2(q,e_1)$, respectively, with $(q,\pi) \in S^2 \times T_q^*S^2$ being the state variables and e_1 is the reference point on S^2. The equation of motion is defined as: $\dot{\omega} = \alpha\left(q \times \log_q(e_1)\right) - \eta(\omega \times q) + (q \times \sigma)\dot{W}$, where, α and μ are system constants, with W being the Wiener process. The kinematic equation is given by $\dot{q} = \omega \times q$.

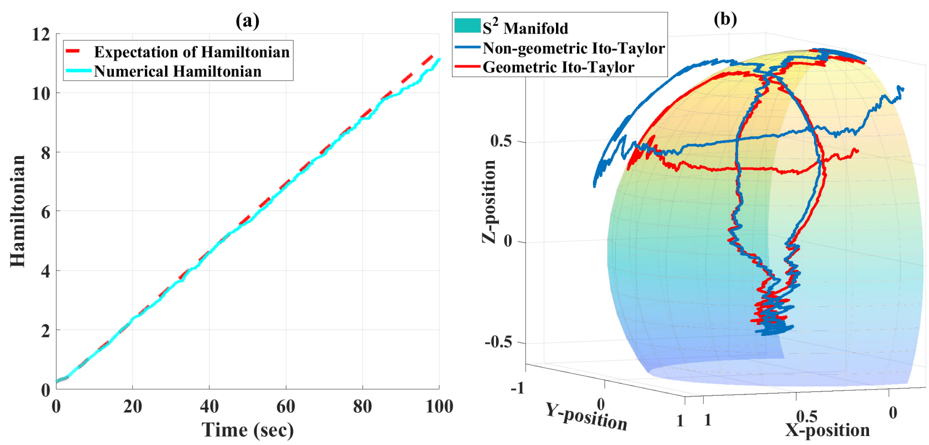

Figure 1: Stochastic DVP Oscillator on S^2: (a) Total Hamiltonian function, (b) Measure of geometry preservation through the response

Results and discussion

The numerical ensemble of the Hamiltonian at each time $H(q_n,\pi_n)$ compared with the analytical solution obtained by taking the expectation of the Hamiltonian $\mathrm{E}[H(q_{n+1},\pi_{n+1})] = \mathrm{E}[H(q_n,\pi_n)] + \langle(q\times\sigma),(q\times\sigma)\rangle \mathrm{d}t$ is presented in Fig. 1(a), which depicts the drift conservation of the Hamiltonian function using the proposed scheme. Fig. 1(b) presents the comparison of geometry preservation of the proposed symplectic geometric preserving scheme with a non-geometric Ito-Taylor-based scheme, where, with the non-geometric integration approach of geometric SDE, the solution drifts away from the manifold. Whereas, the proposed method preserves the geometry of the solution on the manifold. This work presents a new approach for the stochastic non-linear oscillators on manifolds while preserving the manifold's geometry and the Hamiltonian's drift.

References

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