Sufficient conditions to exclude positive Lyapunov Exponents in the Thomas' system

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Abstract. Sufficient conditions to rule out the presence of attractors with positive Lyapunov exponents in the Thomas' system are formulated via the *2-additive compound* of the Jacobian. It will be shown how the problem can be solved both analytically and by using Linear Matrix Inequalities.

Introduction

Ruling out the possibility of chaotic or oscillatory behaviors in dynamical systems has been widely studied and many techniques have been proposed (see, e.g., [1]). A Jacobian-based technique was introduced by Muldowney in the seminal paper [2], in which the author formulates conditions on the matrix norm of the *2-additive compound* of the Jacobian to exclude the existence of nonconstant periodic solutions. Subsequently, the results in [2] has been extended in [3] to rule out periodic and almost periodic solutions. In the same spirit, in [4] it is shown how conditions on the *2-additive compound* of the Jacobian can be also exploited to rule out attractors with positive Lyapunov exponents from a known invariant set $\mathcal{E}$ and how the problem can be traced back to a LMI problem of the form

$$D(P(x)) = \left(\frac{\partial f^{(2)}}{\partial x}(x)\right)^T P(x) + P(x)\left(\frac{\partial f^{(2)}}{\partial x}(x)\right) + \dot{P}(x) \leq 0 \quad \forall x \in \mathcal{E}\ , \tag{1}$$

where $P(x)$ is a properly designed matrix. In this work, we analyze the stability properties of solutions of the Thomas' system applying the results presented in [4].

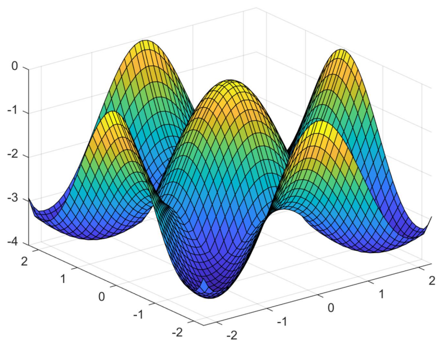

Figure 1: Determinant of $D(P(x))$ within the invariant set $\mathcal{R}$.

Results and discussion

One of the most investigated system in chemical reaction, ecology and evolution is the Thomas' system [5]. It is known that the Thomas' system has a convex and compact invariant set $\mathcal{R} = \{x \in \mathbb{R}^3 : b|x|_\infty \leq 1\}$, where $x = (x_1, x_2, x_3)^T$ is the state vector and b is a positive parameter. Inside the invariant set, the system displays a rich dynamical behavior due to the various bifurcations that occur as b varies in $(0, 1]$. The aim is to find the minimum value b_m of b such that problem (1) can be solved by using the technique in [4]. The value of $b_m = 0.443$ can be obtained both analytically or numerically by solving a LMI problem tuned on only four points of $\mathcal{R}$, which implies that no attractors with positive Lyapunov exponents exist for $b \geq 0.443$. This is confirmed by Figure 1 which reports the function

$$\max_{x_3:(x_1,x_2,x_3)\in\mathcal{R}} \det(D(P(x)))$$

inside the projection of the box $\mathcal{R}$ with respect to the x_3 axis for $b = b_m$, thus showing that the determinant is negative.

References

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