Three large amplitude limit cycles via Zero-Hopf bifurcation from infinity in 3D piecewise linear systems with symmetry.

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Abstract. The existence of the Zero-Hopf bifurcation from infinity is proven for a family of 3D piecewise linear systems with symmetry. Quantitative expressions are provided for the amplitude and period of the bifurcating large amplitude limit cycles living in two and three linearity zones. The simultaneous bifurcation of three limit cycles is shown. The theoretical results are applied to the Bonhöffer-van der Pol electronic oscillator.

Introduction

The zero-Hopf bifurcation for smooth systems has been widely analyzed, see for example [2], but this bifurcation has been rarely studied in piecewise linear differential systems, see [3], where the zero-Hopf bifurcation of a symmetric piecewise linear differential system with three zones in $\mathbb{R}^3$ was characterized obtaining the existence and stability of three periodic orbits bifurcating from the origin and two non trivial equilibria.

In the present work, the family of systems studied has the complex eigenvalues $\sigma(\varepsilon) \pm i\omega(\varepsilon)$, and the real one $\lambda(\varepsilon)$, and the non trivial equilibrium points tend to infinity when ε tends to zero, converting them in zero-Hopf equilibria. These zero-Hopf equilibria give place to a large amplitude bifurcating limit cycle, increasing the catalog of bifurcations at infinity in piecewise linear systems, see [1].

Main results

The piecewise linear differential system considered is written in the generalized Lienard's form defined by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = A_E\mathbf{x} + \mathbf{b}\,\mathrm{sat}(x), \tag{1}$$

where $\mathbf{x} = (x, y, z)^\top \in \mathbb{R}^3$,

$$A_E = \begin{pmatrix} t_E & -1 & 0 \\ m_E & 0 & -1 \\ d_E & 0 & 0 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} t_C - t_E \\ m_C - m_E \\ d_C - d_E \end{pmatrix}. \tag{2}$$

In order to analize the zero-Hopf bifurcation at infinity, we will assume the linear matrix A_E of the external zones has eigenvalues $\lambda(\varepsilon)$, $\sigma(\varepsilon) \pm i\omega(\varepsilon)$, where,

$$\begin{aligned} \lambda(\varepsilon) &= \lambda_1\varepsilon + \lambda_2\varepsilon^2 + O(\varepsilon^3), \\ \sigma(\varepsilon) &= \sigma_1\varepsilon + \sigma_2\varepsilon^2 + O(\varepsilon^3), \\ \omega(\varepsilon) &= \omega_0 + \omega_1\varepsilon + \omega_2\varepsilon^2 + O(\varepsilon^3), \end{aligned}$$

with $\omega_0 > 0$. The main contribution is the following result.

Theorem Consider system (1)-(2) with the above eigenvalue configuration, with $\lambda_1 \neq 0$, $\sigma_1 \neq 0$, $d_C \neq 0$, $\omega_0 > 0$ and define the non-degeneracy parameter $\rho = d_C - \omega_0^2 t_C$. If $\rho \neq 0$, then, for $\varepsilon = 0$ the system undergoes a zero-Hopf bifurcation at infinity, that is, one symmetric limit cycle using the three zones of linearity appears for $\rho\sigma_1\varepsilon > 0$ and ε sufficiently small. In particular, if $\rho < 0$ and $t_C < -\rho(\lambda_1 + \sigma_1)/(\sigma_1\omega_0^2)$, then the limit cycle bifurcates for $\sigma_1\varepsilon < 0$ and is orbitally asymptotically stable.

Furthermore, the period P of the periodic oscillation is an analytic function at 0, in the variable ε, and its series expansion is

$$P = \frac{2\pi}{\omega_0} + \frac{2\pi}{\omega_0}\left(\frac{\sigma_1\left(\omega_0^2 - m_C\right)}{\rho} - \frac{\omega_1}{\omega_0}\right)\varepsilon + O\left(\varepsilon^2\right).$$

The amplitude of the bifurcating limit cycle has the following series expansion,

$$-y_0 = \frac{2\rho}{\pi\sigma_1\omega_0\varepsilon} + O(1).$$

References

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