Fourier Analysis of a Duffing Equation With Delay

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Abstract. We investigate the Duffing-like equation with delay $\ddot{x} + cx(t-T) + x^3 = 0$, where $T > 0$ delay and c is a parameter. In [1], the authors used the method of harmonic balance to discover that this equation displays a remarkable singularity: when T becomes positive, there is a bifurcation in which infinitely many limit cycles with large amplitudes and high frequencies are born. In this work, we extend the techniques used in [1], revealing more complicated dynamics in the delayed Duffing equation. Using a higher order harmonic balance approach allows us to detect the existence of previously unnoticed limit cycles, as well to predict what we have termed a *period splitting bifurcation*.

Introduction

The equation

$$\frac{d^2x}{dt^2} + cx(t-T) + x^3 = 0 \tag{1}$$

has a number of remarkable properties. Here, c is a parameter, and $T \geq 0$ is a delay. In [4], the authors proved that the origin is linearly unstable for all $T > 0$. Later, in [1], the authors predicted that for all $T > 0$, there exist an infinite number of stable limit cycles with very high frequency and large amplitudes. The amplitudes of the limit cycles were determined by using the method of harmonic balance, using the ansatz $x(t) = A\cos(\omega t)$. In subsequent work, including [2] and [3], the authors were able to rigorously prove that the equation supports infinitely many periodic solutions. In particular, in [2, 3], Sah, Fiedler, et. al. were able to find a class of exact solutions by “lifting” certain solutions of the non-delayed Duffing equation to the delay case. We refer the reader to [2] for a more thorough history of progress in understanding this equation.

Results and Discussion

In this work, we investigate equation solutions of (1) using a higher order harmonic balance method, in which we approximate $x(t)$ with a truncated Fourier series of higher order. This extends the results found in [1], allowing us to detect previously unnoticed limit cycles. Our main result is that we are able to predict a qualitative change in the geometry of the limit cycles as T increases, in what we have chosen to term a *period splitting bifurcation.* This bifurcation is characterized by the periodic motion transitioning from oscillating at a single dominant frequency to suddenly oscillating with two component frequencies. See Figure 1 for an illustration. We wish to emphasize how a single computational technique detects such a variety of behaviors of solutions.

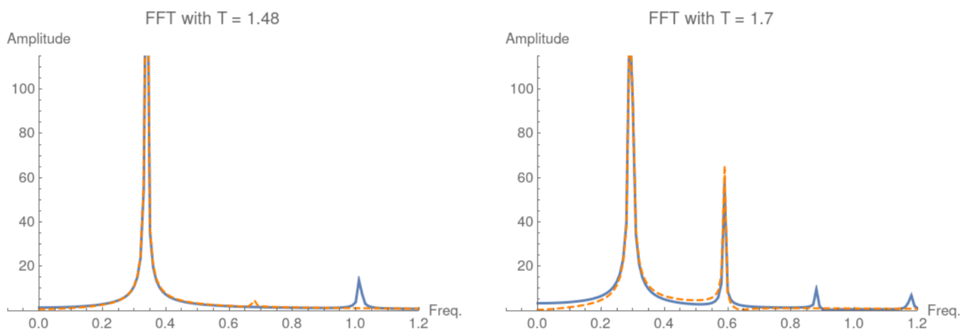

Figure 1: Fast Fourier Transform just before bifurcation (left) and just after (right). Orange dashed is curve predicted by harmonic balance, blue solid is result of numerical integration.

References

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