Existence of a uniform upper bound for the number of limit cycles of planar piecewise linear systems

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Abstract. This presentation is devoted to the open problem of the existence of a uniform upper bound for the maximum number of limit cycles of planar piecewise linear systems with two zones separated by a straight line. We give a positive answer to this question and establish that this number is less than 8.

Introduction

Roughly speaking, the second part of the 16th Hilbert's Problem consists in determining an upper bound for the maximum number of limit cycles of planar polynomial differential systems of degree n. This is one of the most important problems in the analysis of planar differential systems [5], and still remains unsolved even for $n = 2$, the simplest non trivial case.

Here, we consider a version for planar piecewise linear differential systems with two zones separated by a straight line,

$$\dot{\mathbf{x}} = \begin{cases} A_L\mathbf{x} + \mathbf{b}_L, & \text{if} \quad x_1 \leq 0, \\ A_R\mathbf{x} + \mathbf{b}_R, & \text{if} \quad x_1 \geq 0, \end{cases} \tag{1}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $A_{L,R} = (a_{ij}^{L,R})_{2\times 2}$, and $\mathbf{b}_{L,R} = (b_1^{L,R}, b_2^{L,R}) \in \mathbb{R}^2$. For this kind of systems, a limit cycle is defined as an isolated crossing periodic solution.

One of the first works devoted to the study of a uniform upper bound for the maximum number of limit cycles of system (1) is authored by Lum and Chua [7]. There, they conjectured that, under the continuity hypothesis, system (1) had at most one limit cycle. This conjecture was proven in 1998 by Freire et al. [4]. In the literature, there are also partial results about upper bounds for other non-generic families of piecewise linear differential systems but, after more than 30 years since the Lum-Chua's conjecture [7] and hundreds of paper on this subject, the existence of a uniform upper bound for the maximum number of limit cycles that system (1) can have, still remains an open question.

Recently, after obtaining a new integral characterization for *Poincaré half-maps* [1], the authors have considered a new approach to the study of the periodic behavior of piecewise linear systems without the annoying case-by-case study usually needed in previous works. In [2], this approach has been used to give a new simple proof for the Lum-Chua's conjecture. Moreover, the same technique was used in [3] to prove that system (1) has at most one limit cycle when there are not sliding sets in the separation line.

Results and discussion

In this work, by using the integral characterization for Poincaré half-maps provided in [1] and some extensions of the Khovanskiĭ's theory [6], it is proven that there exists a natural bound less than 8 for the number of limit cycles of system (1) .

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