Stochastic basins of attraction for uncertain initial conditions

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Abstract. In this work the problem of stochastic basins of attraction for uncertain initial conditions and deterministic time evolution is addressed. It is shown how it is possible to determine the stochastic basins based on the sole knowledge of the deterministic basins of attraction, and the probability density function of the random initial conditions. The main results are illustrated with a paradigmatic example, the Helmholtz oscillator.

Introduction

The effects of uncertainties over global dynamic structures, such as attractors, basins, and manifolds, are difficult to address. Depending on the nondeterminism in the underlying dynamics, new definitions must be formulated, and the computational difficulty requires specialized algorithms [1]. Previous stochastic basin's definitions are adequate for parameter uncertainty and noise. Still, uncertain initial conditions had been only addressed through integrity measures [2], with the dynamics formulated deterministically.

To consider uncertain initial conditions in a probabilistic framework, it is assumed that the set of initial conditions $\boldsymbol{x}_0$ has a distribution $p(\boldsymbol{x}_0, \boldsymbol{x}, \sigma)$, where σ controls the uncertainty level and $\boldsymbol{x}$ is a possible practical realization of the nominal initial condition $\boldsymbol{x}_0$, and that the time evolution is deterministic. Let A be an attractor of the associated deterministic system, and B_A its deterministic basins of attraction. The stochastic basin of attraction is defined as the function $g_A(\boldsymbol{x}_0, \sigma) \in [0,1]$ that gives the probability that $\boldsymbol{x}_0$ converge towards the attractor A. As an immediate consequence of this definition, we have that

$$g_A(\boldsymbol{x}_0, \sigma) = \int_{B_A} p(\boldsymbol{x}_0, \boldsymbol{x}, \sigma) d\boldsymbol{x}. \tag{1}$$

It is worth to note that this property allows the use of image filtering techniques. Assuming that $p(\boldsymbol{x}_0, \boldsymbol{x}, \sigma)$ is known, the computation starts from the determination of B_A through classical discretization methods (grid of starts, cell-mappings, Ulam method, …), and then $g_A(\boldsymbol{x}_0, \sigma)$ is computed by (1) appositely discretized.

Results and discussion

Initially, the new basin definition is applied to the Helmholtz equation [3],

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^2 = \lambda \sin(\omega t). \tag{2}$$

with initial conditions uniformly distributed over a cell square of dimension s (in pixels). The deterministic basin is obtained through a discretization of the phase-space window $x \in [-0.8; 1.8]$ and $\dot{x} \in [-1; 1]$ into 720x720 cells. Then, a box blur of pixel length s is applied. For increasing s levels, the results show a crescent fuzziness, initially concentrated in the basin boundaries and then spreading over the entire phase-space. Fractal boundaries are more susceptible to uncertainty, becoming fuzzy for lower s levels in comparison to robust regions. As expected, the fuzziness of the stochastic basins of attraction increases by increasing the uncertainty of initial conditions, and the "almost sure" basin $g_A(\boldsymbol{x}_0, \sigma) = 1$ rapidly shrinks.

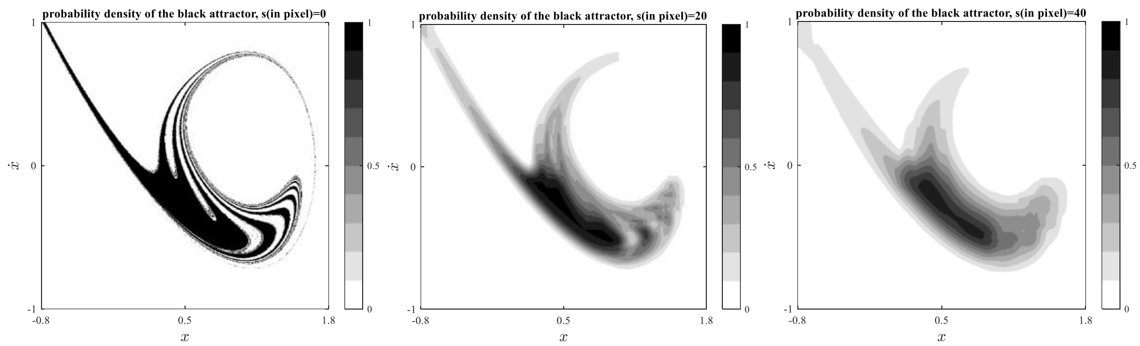

Figure 1: Stochastic basins of attraction for increasing stochastic spreadness s. Resonant attractor. $\alpha = -1, \lambda = 0.07$.

References

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