The Jerk Dynamics of Lorenz Model

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Abstract. The Lorenz model is widely considered as the first dynamical system exhibiting a chaotic attractor the shape of which is the famous butterfly. This similarity led Lorenz to name the *sensitivity to initial conditions* inherent to such chaotic systems, the *butterfly effect* making its model a paradigm of chaos. Nearly thirty years ago, Stefan J. Linz presented in a very interesting paper an "exact transformation" enabling to obtain the jerk form of the Lorenz model and a nonlinear transformation "simplifying its jerky dynamics". Unfortunately, the third order nonlinear differential equation he finally obtained precluded any mathematical analysis and made difficult numerical investigations since it contained exponential functions. In this work, we provide in the simplest way the jerk form of the Lorenz model. Then, a stability analysis of the jerk dynamics of Lorenz model prove that fixed points and their stability, eigenvalues, Lyapunov Characteristics Exponents and of course attractor shape are the exactly the same as those of Lorenz original model.

Introduction

At the very beginning of the sixties, Edward Norton Lorenz (1917-2008), a meteorologist from the famous M.I.T. (Massachusetts Institute of Technology) succeeded in establishing a model for atmospheric convection comprising only three variables. The solution of this weather forecasting model that Lorenz [4] plotted in a three-dimensional phase space is compelled to evolve on a chaotic attractor which resembles the wings of a butterfly. It is probably this form that prompted Lorenz to call the "sensitivity to initial conditions" (described by the French mathematician Henri Poincaré as early as 1908 in his philosophical writings *Science and Method* [5]) the "butterfly effect". During these last two decades, the seminal works of Gottlieb [2] and Sprott [7, 8, 9, 10, 11, 12, 13] have triggered out an increasing interest in the study of chaotic oscillators based on jerk equations, that is, oscillators which can be completely described by third-order ordinary differential equations of the form $\dddot{x} = f(\ddot{x}, \dot{x}, x)$. In 1997, Stephan J. Linz [3] proposed in a very interesting paper an "exact transformation" enabling to obtain the jerk form of the Lorenz model and a nonlinear transformation "simplifying its jerky dynamics". Unfortunately, the third order nonlinear differential equation he finally obtained precluded any mathematical analysis and made difficult numerical investigations since it contained exponential functions. Let's notice that the jerk form in x of the Lorenz model that we will provide below is exactly the same as those obtained by Linz but presented in a different way. In 2014, Buscarino *et al.* [1] used *linear combinations* of the three nonlinear ordinary differential equations modeling the Chua's circuit to deduce its jerk forms in x and z. Recently, Xu and Cao [14] proposed to use the so-called *controllable canonical form* to provide all the jerk forms dynamics of Chua's circuit. In this paper, following the method of *linear combinations* proposed by Buscarino *et al.* [1], we provide the jerk form in x of Lorenz model. Thus, by making a comparison of fixed points and their stability, eigenvalues, Lyapunov Characteristic Exponents and attractor shapes between the original three-order Lorenz model and its first jerk form in x we demonstrate the topological equivalence of both systems.

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