Optimal projection in a Koopman-based sorting-free Hill method

Fabia Bayer, Remco I. Leine
Institute for Nonlinear Mechanics, University of Stuttgart, Germany
ORCID 0000-0002-2038-3374 and ORCID 0000-0001-9859-7519

Abstract. In this work, the performance of a novel method to determine the stability of periodic solutions based on the Hill matrix is examined. Using the Koopman framework, the linear time-periodic perturbed dynamics around a periodic solution can be approximated by a linear autonomous system of higher order, whose system matrix is the well-known Hill matrix. The monodromy matrix can hence be approximated by the Hill matrix, using only a matrix exponential and a projection. This projection is not uniquely determined, and various ways to obtain a suitable projection are discussed in this paper. The numerical efficiency of the novel method is illustrated for the vertically excited multiple pendulum.

Introduction

The numerical characterization of periodic solutions and their stability in nonlinear systems is a task of great interest in engineering application. One common approach to find these periodic solutions is the Harmonic Balance method (HBM). The HBM itself does not give stability information, however the Hill matrix $\mathbf{H}$ can be found and constructed easily. As its eigenvalues approximate the Floquet exponents, this can be used as a stability criterion. However, in a so-called *sorting* process, only a nontrivial subset of these eigenvalues must be considered for correct assertion of stability. This is an ongoing area of research [1, 2].
Recently, the authors proposed a Koopman-based stability method [3]. For the perturbed dynamics $\mathbf{y}$, the dynamics of the Koopman basis $\mathbf{\Psi}(\mathbf{y},t) = [N_{\mathbf{u}} i\omega t, \ldots, -N_{\mathbf{u}} i\omega t]^{\mathrm{T}} \otimes \mathbf{y}$ are approximated by

$$\dot{\mathbf{\Psi}} \approx \dot{\mathbf{z}} = \mathbf{H}\mathbf{z}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{\Psi}(t) \approx \mathbf{C}(t)\mathbf{z}(t) = \mathbf{C}(t)e^{\mathbf{H}t}\mathbf{W}\mathbf{y}_0 \ ,$$

yielding $\mathbf{C}(T)e^{\mathbf{H}T}\mathbf{W}$ as an approximation of the monodromy matrix $\mathbf{\Phi}_T$. Its n eigenvalues, the Floquet multipliers, carry the stability information. The projection $\mathbf{C}$ from the linear autonomous Koopman lift to the monodromy matrix is not unique and influences the accuracy of the determined Floquet multipliers.

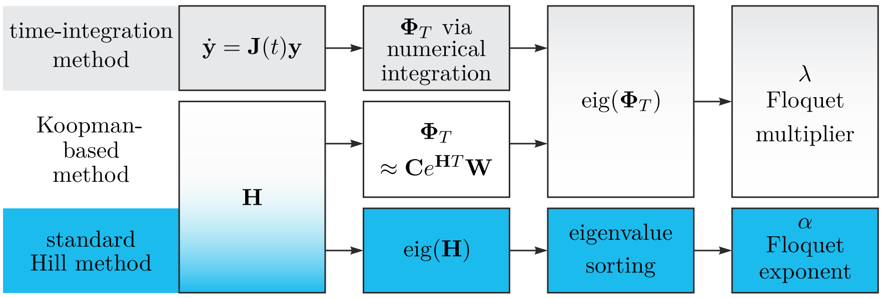

Figure 1: Flowchart comparing the three general stability approaches for periodic solutions.

Results and discussion

Application of the above method to a vertically excited multiple pendulum as a generalization of the Mathieu equation shows that the choice of the projection matrix $\mathbf{C}$ in the presented method greatly influences accuracy and convergence speed. If $\mathbf{C}$ is chosen to be constant and picking the middle rows of $\mathbf{\Psi}$, convergence to the correct Floquet multipliers is observed. The convergence rate can, however, be improved by other choices. In particular, evaluation of the approximated perturbed dynamics over one period using an integral or a low number of time samples yields a projection matrix that improves this convergence rate. However, both these techniques can exhibit numerical issues. If the maximum considered frequency $N_{\mathbf{u}}$ is large, the quadratic program needed for the integral approach tends to stall due to a numerically semidefinite or even indefinite cost matrix. Numerical rank loss can also occur while solving a linear equation system of time samples, again preventing the ideal $\mathbf{C}$ from being found. However, this rank loss can be circumvented by using more or differently spaced time instants. For the multiple pendulum, structure in these optimal solutions can be used to explicitly specify a $\mathbf{C}$ matrix for larger but similar systems, circumventing these numerical problems.

References

[1] Zhou J., Hagiwara T., Araki, M. (2004) Spectral characteristics and eigenvalues computation of the harmonic state operators in continuous-time periodic systems. *Systems & Control Letters* **53(2)**: 141–155
[2] Lazarus, A., Thomas, O. (2010) A harmonic-based method for computing the stability of periodic solutions of dynamical systems. *Comptes Rendus Mécanique* **338(9)**: 510–517
[3] Bayer, F., Leine, R.I. (2022) Sorting-free Hill-based stability analysis of periodic solutions through Koopman analysis. *ResearchSquare [Preprint], submitted to Nonlinear Dynamics*, https://www.researchsquare.com/article/rs-2183060/v1