# Operator Splitting in the Finite Element Analysis of Fokker-Planck Equations

**Hangyu Fu**[*], Lawrence A. Bergman[**], D. Michael McFarland[*], Xiangle Cheng[*], and Huancai Lu[*]

[*]*College of Mechanical Engineering,, Zhejiang University of Technology, Hangzhou, China*

[**]*Aerospace Engineering Department, University of Illinois at Urbana-Champaign, Urbana, Illinois, USA*

**Abstract**. The Fokker-Planck equation governing the probability density function (pdf) of a mechanical system can be readily discretized in state space using finite element analysis. However, even low-dimensional systems, such as a single-degree-of-freedom Duffing oscillator with two states, lead to matrix equations of large dimension, and these systems of equations grow rapidly as the number of states increases. If multiple parameter values or initial conditions are to be considered, computation of the nonstationary response by standard integration techniques rapidly becomes impractical. We examine here the use of operator splitting, where the finite element matrices are formulated separately for the convection and diffusion terms in the FP equation, leading to a time marching scheme based on the resulting state transition matrices. It is found that the pdf can be computed much more efficiently using this approach than with, for example, an adaptive Runge-Kutta algorithm.

## Introduction

The nonstationary probability density function (pdf) of a mechanical system is governed by the Fokker-Planck (FP) equation, a partial differential equation (PDE) which is readily obtained from a state-variable representation of the original equation of motion. This PDE can be discretized in state space using, for example, a Galerkin finite element formulation, resulting in an equation of the form $\mathbf{M}\dot{\mathbf{p}}(t) + \mathbf{K}\mathbf{p}(t) = \mathbf{0}$, subject to the initial conditions $\mathbf{p}(0) = \mathbf{p}_0$. It is typically found that 100 elements are needed in each dimension of the state space to obtain a stable, accurate solution, resulting in approximately 10,000 degrees of freedom in the discrete problem. The problem size grows exponentially with the number of states in the system, and calculations for 2-degree-of-freedom (DOF) systems with four states (two displacements and two velocities) generally remain impractical in most applications (such as in design, where repeated solutions are required).

We have applied operator splitting to this problem by separating the convection and diffusion terms in the FP equation and discretizing them separately to produce two matrices, $\mathbf{K}_1$ and $\mathbf{K}_2$, whose sum replaces $\mathbf{K}$ in the equation above. These are found to be much better conditioned than the original matrix; as a result, a state transition matrix (STM) can be computed for each of them. These are used to advance the solution by fixed time steps, using either a composite STM or a more accurate (Strang splitting [1]) algorithm.

## Results and discussion

Preliminary results obtained with this approach are very promising. As an example, we consider the single-DOF Duffing oscillator with negative linear stiffness studied by Spencer and Bergman [2]. Figure 1 compares a cross section of the stationary pdf to the exact solution, and shows the numerical solution for the pdf at a point computed with and without splitting. In this example, running on an x86 (notebook) processor, computation of the response for 4 linearized natural periods using an adaptive Runge-Kutta algorithm required 548.8 minutes; with operator splitting, this was reduced to 4.6 minutes.

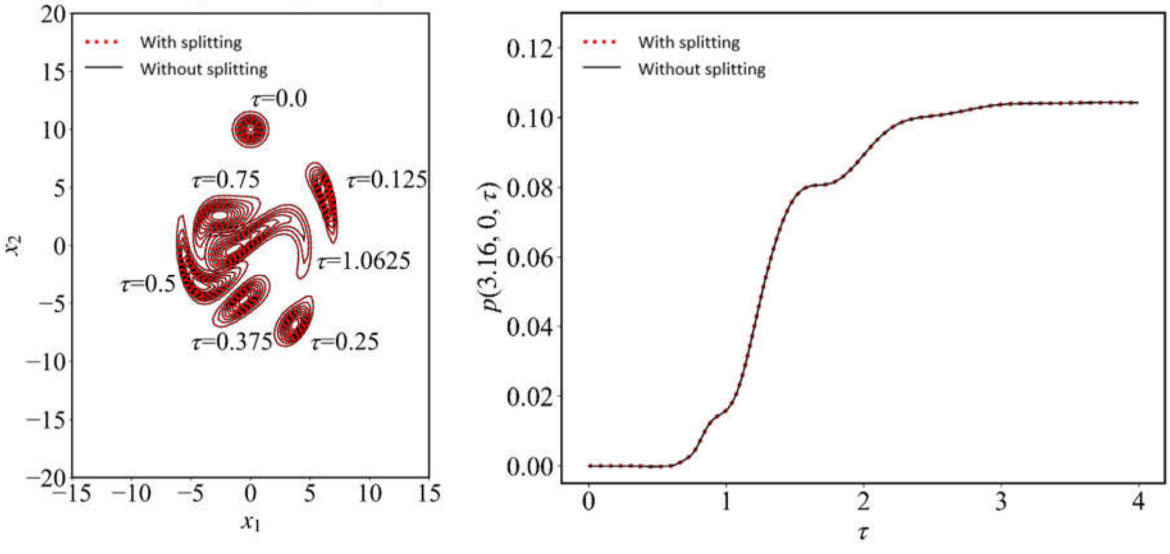


Figure 1: Results for the Duffing oscillator: (left) superimposed contour plots of the nonstationary pdf at selected times; (right) growth over time of the pdf at the location of a local maximum stationary value.

## References

[1] McLachlan R. I. and Quispel G. R. W. (2002) Splitting Methods. *Acta Numerica* **11**:341–434.

[2] Spencer B. F. and Bergman L. A. (1993) On the Numerical Solution of the Fokker-Planck Equation for Nonlinear Stochastic Systems. *Nonlinear Dynamics* **4**(4):357–372.