Small In-plane Oscillations of a Slack Catenary by the Rayleigh-Ritz method

Bidhayak Goswami*, Indrasis Chakraborty** and Anindya Chatterjee*

**Department of Mechanical Engineering, IIT Kanpur, India*

***Lawrence Livermore National Laboratory, US Department of Energy*

Abstract. We investigate small in-plane motions of a slack catenary using the Rayleigh-Ritz method. Using assumed modes and the Lagrangian approach, linearized equations with one holonomic constraint yield natural frequencies that match simple experiments. However, if the mode shapes thus obtained are themselves used as assumed modes, the approximation fails. A formulation that acknowledges *nonlinear* normal modes eliminates the constraint and gives the correct frequencies again. This problem offers interesting insights into approximations, constraints, and linearization.

Introduction

A *catenary* is an inextensible chain that hangs between two fixed ends. Its equilibrium shape is a hyperbolic cosine. Its in-plane oscillations have been studied by many authors before [1-5], using various approximations. The difficulty in directly using assumed modes in a Lagrangian formulation (i.e., the Rayleigh-Ritz method) is that the chain has a nonlinear pointwise inextensibility constraint involving spatial derivatives of the assumed modes. We cannot easily express the displacements of the chain in both horizontal and vertical directions using a complete basis of kinematically admissible functions. However, simple approximations are possible if we are willing to do numerical integration in space to obtain various coefficients within the Lagrangian.

We begin by approximating the vertical motion as $v(x,t) = \sum_{i=1}^{N} q_i(t)\phi_i(x)$, where the $\phi_i(x)$ are zero at the endpoints of the catenary. Pointwise inextensibility yields a series expansion for the spatial derivative of the horizontal displacement, i.e., $u_x(x,t)$. Integrating u_x from one endpoint gives a u that need not be zero at the other endpoint. This introduces a single scalar holonomic constraint on the q_i's. Now implementing the Lagrangian formulation, we note that the potential energy in the Lagrangian is *linear* in the generalized coordinates and cannot cause oscillations. The Lagrange multiplier corresponding to the holonomic constraint plays that role. We obtain static equations that determine the Lagrange multiplier, and dynamic equations that use that multiplier value to yield natural frequencies and mode shapes as in Fig. (1).

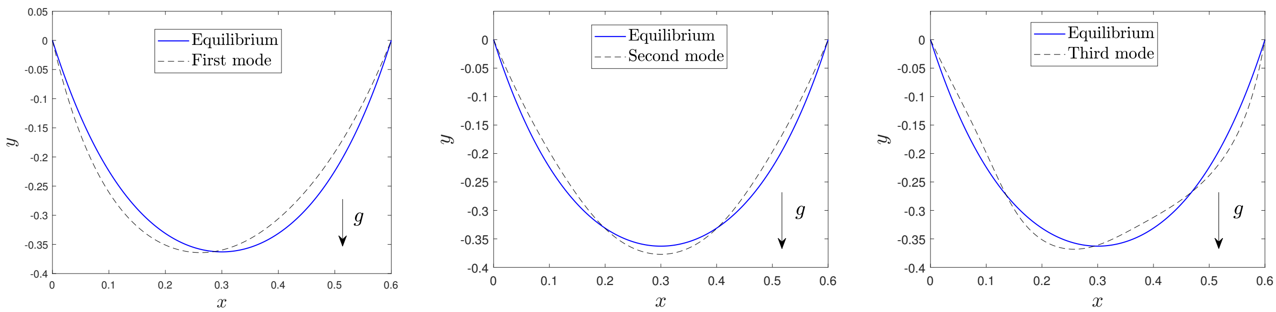

Figure 1: In-plane mode shapes of a slack catenary.

We now come to a puzzle. When the modes shapes determined above are reused in a fresh assumed modes calculation, the approach fails. First, the static equations used to find the Lagrange multiplier disappear; yet that multiplier value determines the natural frequencies. Second, the nonlinear constraint equation suggests that nonzero motion is impossible. Resolution lies in noting that the oscillations are along nonlinear normal modes. Deviations from the eigenspace must be allowed. Incorporating the deviation in a fresh assumed modes expansion and eliminating the holonomic constraint, we remove the Lagrange multiplier, introduce nonlinearity in the potential energy, and recover the correct natural frequencies and mode shapes.

Our study offers several interesting insights into Lagrangian mechanics. It also, for the first time to our knowledge, demonstrates use of the Rayleigh-Ritz method for this classical problem.

References

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