On the role of wave resonances in the nonlinear dynamics of discrete systems

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Abstract. Nonlinear systems are driven by the interplay between modes, intimately related to whether combinations in wavenumbers and akin in frequency are allowed to interact. Our study investigates the role played by resonances in two selected models: FPUT and Metamaterials, combining dynamical-systems methods with high-accuracy numerical simulations. In each case, we establish a leading order of interactions that dominate the behaviour of the system.

The Fermi-Pasta-Ulam-Tsingou (FPUT) chain was a pioneer work in discrete modellization. The original problem considered a one-dimensional chain made out of N identical masses connected by nonlinear springs with fixed boundary conditions (BCs). Against the expectations, the system seemed not to thermalize: the chain displayed the phenomenon of recurrence, which led to the discovery of solitons and integrable systems. Later it was shown that thermalization does occur over very long times. Our work looks at the $\alpha + \beta$ FPUT system from a fresh viewpoint: allowing for periodic BCs gives rise to several non-trivial resonances, unlike the case of fixed BCs. For our system, it is known that 3-wave resonances do not exist, whereas 4-wave resonances exist but are completely integrable. Thus, one must look at the next interaction orders, namely 5-wave and 6-wave resonances, in order to get energy mixing across modes. A second theme concerns the analytical and numerical treatment of a model of metamaterials based on the elementary FPUT chain coupled with resonators. Metamaterials are artificially engineered geometrical structures that show preference for certain wave excitations, thus encountering the increasing demands for specific features that traditional materials cannot satisfy. The model has a far richer resonant manifold than FPUT. For instance, there is a region of parameter space where 3-wave resonances exist.

Results and discussion

We identified all FPUT resonant quintets and sextuplets analytically [1]. The current work focuses on the dynamical effects of these resonances, using numerical simulations backed with analytical normal-form transformations [2]. Preliminary results (Fig. (a)−(b)) confirm the relevance of quintets, in scenarios ranging from quasi-integrable quintets to hyperchaos due to interacting quintets. We show numerically that these interactions dominate important aspects: the evolution towards thermalization, the distribution of chaotic attractors, and the distinction between fixed and periodic BCs regarding integrability. In the case of metamaterials we show that resonant triads are responsible for the complex dynamics that takes the system beyond integrable quartets, leading eventually to ergodic behaviour and energy equipartition (Fig. (c)−(d)).

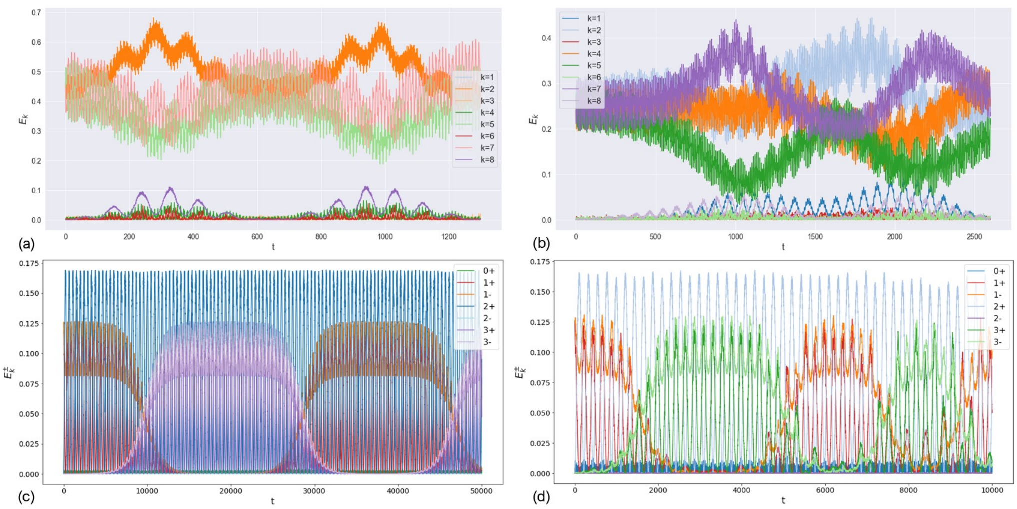

Figure 1: Time evolution of the energy in each normal mode in a series of high-accuracy simulations. FPUT ($N = 9$) shows a quasi-integrable resonant quintet (a) and a chaotic regime of two interacting resonant quintets (b). Metamaterial model ($N = 4$) shows an off-resonance regime (c) versus a chaotic resonant triad regime (d).

References

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