The blow-up method applied to monodromic singularities of the plane

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Abstract. The blow-up method proved its effectiveness to characterize the integrability of the resonant saddles giving the necessary conditions to have formal integrability and the sufficiency doing the resolution of the associated recurrence differential equation using induction. In this work we apply the blow-up method to monodromic singularities in order to solve the center-focus problem. The case of nondegenerate monodromic singularities is straightforward since any real nondegenerate monodromy singularity can be embedded into a complex system with a resonant saddle. Here we apply the method to nilpotent and degenerate monodromic singularities solving the center problem when the center conditions are algebraic.

Introduction

A monodromic nondegenerate singular point placed at the origin of a differential system on $\mathbb{R}^2$ takes the form

$$\dot{u} = v + P(u,v), \qquad \dot{v} = -u + Q(u,v), \tag{1}$$

where $P(u,v)$ and $Q(u,v)$ are real analytic functions without constant and linear terms. Such singular point is a center, if and only if, the system has a first integral of the form $\Phi(u,v) = u^2+v^2+\sum_{k+l\geq 3}\phi_{kl}u^k v^l$, analytically defined around it. Therefore the center-focus problem reduces to prove the existence of such analytic first integral. We can complexify system (1) defining the complex variable $x = u+iv$ and system (1) is transformed to the equation $\dot{x} = ix + R(x,\bar{x})$. Considering its complex conjugate equation and defining $y := \bar{x}$ as a new variable and $\bar{R}$ as a new function we obtain the complex system $\dot{x} = x + G(x,y)$, $\dot{y} = -y + H(x,y)$ after the time scaling $idt = dT$. The above power series becomes now $\Psi(x,y) = xy + \sum_{i+j>2}\psi_{ij}x^i y^j$, satisfying $\dot{\Psi} = \sum_{i=1} v_{2i+1}(xy)^{2i+2}$, where v_{2i+1} are polynomials in the parameters of the system. We note that if all the polynomials v_{2i+1} vanish then the power series $\Psi(x,y)$ becomes a first integral of the system. The singular point at the origin is now a $1:-1$ resonant saddle singular point and the values v_{2i+1} are the *saddle constants*, see [2, 3]. The $1:-1$ resonant saddle singular point is generalized into the $p:-q$ resonant saddle singular point which placed at the origin the differential system is the form $\dot{x} = p\,x + F_1(x,y)$, $\quad \dot{y} = -q\,y + F_2(x,y)$, where F_1 and F_2 are analytic functions without constant and linear terms with $p, q \in \mathbb{Z}$ and $p, q > 0$, see [1, 3] and references therein. In this case a $p:-q$ resonant saddle singular point is called a resonant center, if an only if, there exists a meromorphic first integral $\Psi = x^q y^p + \sum_{i+j>p+q}\psi_{ij}x^i y^j$ around it.

Results and discussion

The blow-up method to detect formal integrability works performing the blow-up $(x,y) \to (x,z) = (x,y/x)$, So that the origin is replaced by the line $x = 0$, which contains two singular points that correspond to the separatrices, and study if one of these two resonant saddle points is orbitally analytically linearizable. Finally to prove the sufficiency of the original system we can apply the following result.

Theorem 1 *Assume that we have proved that system in variables (x,z) has a formal first integral $\tilde{\mathcal{H}}(x,z)$ then if the function $\tilde{H} = \tilde{\mathcal{H}}(x,y/x)$ is well-defined at the origin of system in the variables (x,y) them is analytic integrable around it.*

Summarizing we study the connected singular points at infinity and if they are formally integrable and the first integral can be extended up to the origin then the origin is also formally integrable. Here we apply the method to nilpotent and degenerate monodromic singularities in order to solve the center-focus problem. For degenerate monodromic singularities there is no method to approach the center-focus problem. The method shows that the formal integrability of the points at infinity is intimately linked with the center problem at the origin even though the center at the origin be non formally integrable. The method determine center conditions for monodromic singularities which are algebraically solvable. We solve several non trivial examples.

References

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