Modeling stick balancing with stochastic delay differential equations

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Abstract. In this study, we present an extended model for stick balancing. The control parameters in the PD controller are stochastic variables, and we investigate the properties of the corresponding stochastic delay differential equation (SDDE). The noise changes the stability boundaries and reveals important dynamics otherwise hidden in deterministic models. We can mitigate the differences between the popular deterministic models and measurements with this approach.

Introduction

Researchers investigate the neural control of humans through various balancing tasks, for example, postural balance or stick balancing. A popular model of stick balancing is the inverted pendulum and cart system, shown in Fig. 1a), which is unstable. However, we can stabilize it with an appropriately chosen control force. PD control is a common choice [1] because we assume that humans try to measure the angle and the angular velocity. The results of the PD controllers are promising, but we run into significant differences when comparing them to measurement outcomes. Specifically, we are interested in the critical length of the stick, which is the length we can still stabilize with a given reflex delay. There is an analytical formula for the critical length as a function of the delay based on the deterministic model, namely $l_{\mathrm{cr,det}} = 3g\tau^2/4$. However, we observe a shift between the measured and analytical curves in Fig. 1b). In reality, the balancing process is more unstable than the deterministic model. There are some extensions to improve the model, for example, sensory deadzones [2] or other controllers [3]. However, we find that handling the parameters of the PD controller as stochastic variables favorably impacts the dynamical behavior. We investigate the second moment, which tells us about the typical deviations of the trajectories from the deterministic solution. Second moment instability means large variances: it translates to uncontrollable angles and angular velocities in our stick-balancing case.

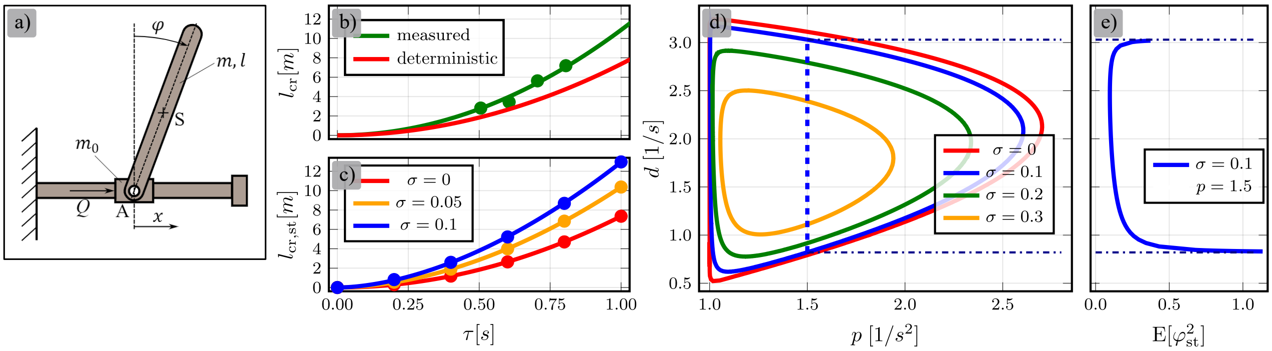

Figure 1: a) Mechanical model. b) Measured critical length l_{cr} as a function of the reaction delay τ. c) Critical length as a function of the delay utilizing the stochastic model with different noise intensities σ. d) Second moment stability boundaries of the stochastic model. e) Stationary second moment dynamics $\mathrm{E}[\varphi^2_{\mathrm{st}}]$ along $p = 1.5$.

Results and discussion

In Fig. 1d), we present the stability boundaries of the stochastic model in the control parameters' plane. We see the effect of the noise intensity at a given stick length and delay: Larger noise results in smaller stable regions. We determine the critical length at a given delay for different noise intensities by selecting the value where the stable region disappears. Fig. 1c) shows the stochastic critical length curves as a function of the delay. The curves are still second-order polynomials, but their multiplier is a function of the noise intensity, namely $l_{\mathrm{cr,st}} = \gamma(\sigma) l_{\mathrm{cr,det}}$. Based on this, the estimation of the noise level is around 5% in our measurements. However, in simulations, the highest noise intensity providing stable solutions was 4%. Fig. 1e) shows the stationary second moment dynamics, and we observe high variances even in the stable region. If these variances reach the limit angle of 90°, the probability of losing balance increases significantly. Therefore, we modify our *stable balancing* condition from the disappearance of the stable region to a limit of the stationary second moment. We hope this approach brings the measurements and simulations even more in agreement.

References

[1] Tamas Insperger, John Milton. (2014). Sensory uncertainty and stick balancing at the fingertip. Biological cybernetics. 108. 10.1007/s00422-013-0582-2.

[2] Tamas Insperger, John Milton (2017) Stick Balancing with Feedback Delay, Sensory Dead Zone, Acceleration and Jerk Limitation. *Procedia IUTAM* **22**:59-66.

[3] Zhang L, Stepan G, Insperger T. 2018 Saturation limits the contribution of acceleration feedback to balancing against reaction delay. J. R. Soc. Interface 15: 20170771.