Geometric Ito-Taylor Weak 3.0 integration scheme for dynamical systems on manifolds

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Abstract. The understanding of the interaction of stochasticity and nonlinearity is a key aspect of modeling complex dynamical systems. The dynamics of physical systems with inherent uncertainties naturally evolve on a mathematical structure defined as a manifold. Towards this, the present work proposes a new higher order geometry preserving stochastic integration scheme based on Ito-Taylor expansion for stochastic dynamical systems evolving on the manifold. This study blends the concepts of Ito calculus and the theory of Lie groups. The preservation of the geometry of the manifold is ensured by exploiting the relation between Lie group and Lie algebra through the exponential map.

Introduction

Naturally occurring systems often have inherent randomness rendering their analysis difficult. Stochastic Differential Equations (SDEs) facilitate the accurate modeling of such dynamical systems. The application of SDEs for dynamical systems evolving on the Euclidean space is well established [1]. However, most of the physical systems evolve on manifolds, resulting in the analysis of such systems quite difficult since the dynamics is constrained by certain geometrical constraints [2]. Although several literature have developed integration schemes for dynamical systems on manifold, they are mostly centered around deterministic dynamics. Literature [2] developed a stochastic Magnus expansion based geometric Euler-Maruyama (g-EM) scheme for solving geometric SDEs. However, such lower order schemes are not suitable for solving nonlinear SDEs [1]. Towards this, a new higher order geometric stochastic integration scheme is developed for solving nonlinear geometric SDEs such that both stochasticity is taken into account as well as the geometry of the manifold is preserved. Consider a geometric SDE on the manifold as, $d\mathbf{X}\left(t\right) = \mathbf{A}\left(\mathbf{X},t\right)\mathbf{X}\left(t\right)dt + \sum_{r=1}^{d}\mathbf{B}_r\left(\mathbf{X},t\right)\mathbf{X}\left(t\right)dW_r\left(t\right)$, where, $\mathbf{X}\left(t\right)$ is the stochastic process and $dW_r\left(t\right)$ denotes the increments of Wiener process. Considering a stochastic Magnus expansion expansion [2], the solution is obtained as, $\mathbf{X}\left(t\right) = \exp\left(\boldsymbol{\Omega}\left(t\right)\right)\mathbf{X}_0$, where, $\mathbf{X}\left(0\right) = \mathbf{X}_0$. $\boldsymbol{\Omega}\left(t\right)$ is a matrix defined on the Lie algebra (space of skew-symmetric matrices) of the manifold such that an SDE corresponding to $\boldsymbol{\Omega}\left(t\right)$ can be written as, $d\boldsymbol{\Omega}\left(t\right) = \boldsymbol{\alpha}\left(\mathbf{A},\mathbf{B}_r\right)dt + \sum_{s=1}^{d}\boldsymbol{\beta}_s\left(\mathbf{B}_r\right)dW_s\left(t\right)$ with $\boldsymbol{\Omega}\left(0\right) = 0$. Since the Lie algebra is analogous to the Euclidean space, following the development in [1], the geometric Weak 3.0 mapping can be constructed for solving the aforementioned SDE on the Lie algebra.

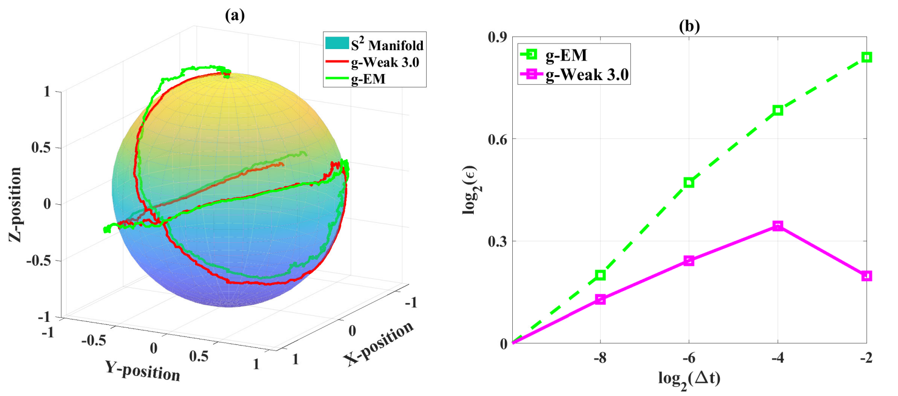

Figure 1: (a) Measure of geometry preservation, (b) Comparison of global error for stochastic Keplerian oscillator on S^2.

Results and Discussions

Fig. 1(a) shows the response trajectory of the Keplerian oscillator on the sphere S^2 for a time step of $\Delta t = 0.01$s. It is clearly observed that the solution using non-geometric Weak 3.0 scheme [1] drifts away from the surface of the manifold, thus failing to preserve the geometry, whereas, the proposed geometric Weak 3.0 (g-Weak 3.0) scheme preserves the geometry for coarser time step. Fig. 1(b), shows that the global error for g-EM increases for coarser time step whereas, the global error corresponding to g-Weak 3.0 is significantly less, thus reducing the computational time. The global error is defined as, $\log_2\left(\max\left(E\left[X_{ref}\right]\right) - \max\left(E\left[X\right]\right)\right)$.

References

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