Kink formation and propagation in a nonlinear, one–dimensional, infiltration model

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Abstract. Analytical and numerical kink solutions to a one–dimensional model of infiltration, granular flows, water flows in unsaturated soils, and disordered ensembles of droplets in microfluidic channels at low Reynolds number are presented. The model is governed by a time–dependent nonlinear advection–diffusion equation where the advective flux is a quadratic function of the flow variable and the diffusion coefficient is the sum of a constant plus a monomial nonlinear function of the absolute value of the flow variable gradient. It is shown the sharp kinks appear when the diffusion is degenerate, and the kink's thickness increases as the diffusion coefficient is decreased.

Introduction

Many models of water flow in unsaturated soils [1], granular flows, infiltration under steady rainfall, motion of droplets in microfluidic channels [2], etc., are based on nonlinear, one–dimensional, advection–diffusion equations, where the diffusion coefficient usually depends on the gradient of the flow variable and may be degenerate, i.e., it may become zero [3]. In the presence of degeneracy, the kink or shock wave solutions provided by these models may result in sharp kinks characterized by very steep kinks which exhibit fast transitions to the upstream and downstream boundary conditions.

In this work, a one–dimensional, infiltration model is presented. The model has a quadratic advective flux and a nonlinear diffusion one, and the diffusion coefficient is the sum of a constant plus a monomial nonlinear function of the absolute value of the flow variable gradient. The nonlinear part of the diffusion coefficient increases (decreases) as the flow gradient increases when the power of the monomial is larger (smaller) than one, and the diffusive flux becomes a Heaviside function when the power of the monomial is exactly zero; the power of the monomial cannot be less than zero because, otherwise, the diffusive flux becomes infinite if the gradient of the flow variable becomes nil. Analytical solutions of the kink, topological soliton or shock wave type to the one–dimensional model proposed here are reported for powers of the nonlinear contribution to the diffusion coefficient equal to 0, 1/2, 1 and 2. Numerical solutions are also obtained for both other powers of the nonlinear diffusion coefficient and several initial conditions.

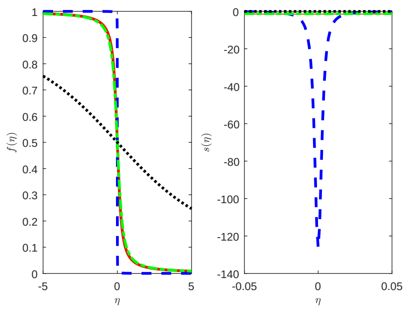

Figure 1: Kink solutions. (Left: flow variable; right: gradient of the flow variable.)

Results and discussion

Some sample results illustrating the kinks for several values of the coefficients that characterize the linear and nonlinear contributions to diffusion are presented in Figure 1. Figure 1 (ledt) shows the flow variable and illustrates the increase in the kink's steepness as the coefficient of the nonlinear contribution of the diffusion coefficient is decreased. For large values of this coefficient, the kink is very thick and the flow gradients are small, whereas, for small values, the kink is very thin and the flow gradients are very large and localized. The flow gradients are related to the strain rate; therefore, a decrease of the diffusion coefficient results in an increase in the flow strain as shown in Figure 1 (right), while a zero diffusion coefficient results in the formation of a singular shock wave in finite time if the upstream boundary condition for the flow variable is larger than its downstream value. Numerical simulations have been performed to determine the effect of several initial conditions on wave propagation and the time required for the formation of kinks, and indicate that the kink–formation formation is larger for piecewise initial conditions than for those of the hyperbolic tangent type.

References

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