Study of a double-zero bifurcation in a Lorenz-like system. Application to the analysis of the Lorenz system

A. Algaba*, M.C. Domínguez-Moreno*, M. Merino* and **A.J. Rodríguez-Luis****

*Departamento de Ciencias Integradas, Centro de Estudios Avanzados en Física, Matemática y Computación, Universidad de Huelva, Spain.

**Departamento de Matemática Aplicada II, E.T.S. Ingeniería, Universidad de Sevilla, Spain

Abstract. In this work we consider a Lorenz-like system and study the double-zero bifurcation it exhibits. The local study of the double-zero bifurcation provides partial results that are extended by means of numerical continuation methods. Specifically, a curve of heteroclinic orbits is detected. The degeneracies exhibited by this global connection guarantee the presence of a very rich dynamical behavior. Finally, the results obtained allows to explain the origin of the curves of homoclinic and heteroclinic connections related to the T-point-Hopf bifurcation exhibited by Lorenz system.

Introduction

We consider the system

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = \rho x - y - xz, \qquad \dot{z} = -bz + xy + Dz^2, \tag{1}$$

where $D \in \mathbb{R}$, so that the Lorenz system

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = \rho x - y - xz, \qquad \dot{z} = -bz + xy, \tag{2}$$

is embedded in (1) when $D = 0$. System (1) is also invariant under the change $(x, y, z) \to (-x, -y, z)$. The origin E_1 of system (1) exhibits a double-zero bifurcation [1] (a double-zero eigenvalue with geometric multiplicity two), for $\rho = 1$, $b = 0$, $\sigma \neq -1$, $D \neq 0$, in which a second equilibrium $E_2 = (0, 0, b/D)$ is also involved. Our theoretical analysis demonstrate the existence of transcritical, pitchfork and Hopf bifurcations of equilibria as well as a heteroclinic cycle between E_1 and E_2. Moreover, a degenerate double-zero bifurcation occurs when $\sigma = 1/3$.

Results and discussion

By means of numerical continuation methods, the local results can be extended and applied to the study of (1) when $D \neq 0$ (see Figs. 1(a)-(c)). In this way we find several degeneracies in the heteroclinic connections that lead to complex dynamical behavior (some of them are even of codimension three). For instance, in the vicinity of one of these degenerate heteroclinic cycles we conjecture the existence of an infinite sequence of bifurcation curves of various types that emanate from the corresponding point in the parameter plane: saddle-nodes of asymmetric and symmetric periodic orbits, period-doublings of the asymmetric periodic orbits, symmetry-breakings of the symmetric periodic orbits, homoclinic connections of the origin..., which implies the existence of diverse types of attractors in a neighborhood of the origin. On the other hand, when we decrease the value of D until reaching $D = 0$, our study allows to see how the global connections related to the double-zero bifurcation of system (1) give an explanation of the origin of the curves of homoclinic and heteroclinic connections related to the T-point-Hopf bifurcation exhibited by Lorenz system (see Fig. 1(d)) [2, 3].

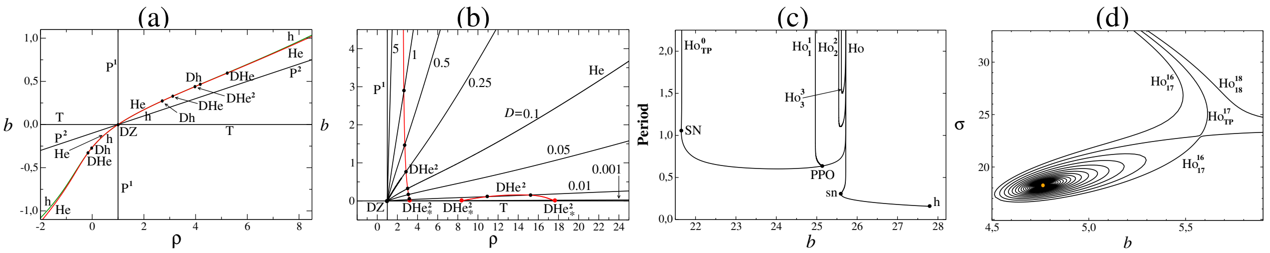

Figure 1: (a) Partial bifurcation set for $\sigma = 3$, $D = 0.1$. (b) Curves of nondegenerate heteroclinic cycles related to the double-zero degeneracy for several values of D. (c) Bifurcation diagram of periodic orbits related to several homoclinic connections, for $\rho = 50$, $D = 0.21$, $\sigma = 100$. (d) Curves of homoclinic connections related to a T-point heteroclinic loop when $\rho = 50$, $D = 0$ (Lorenz system).

References

[1] A. Algaba, M.C. Domínguez-Moreno, M. Merino, A.J. Rodríguez-Luis, Double-zero degeneracy and heteroclinic cycles in a perturbation of the Lorenz system, Commun. Nonlinear Sci. Numer. Simul. 111 (2022) 106482.

[2] P. Glendinning, C. Sparrow, T-points: a codimension two heteroclinic bifurcation. J. Stat. Phys. 43 (1986) 479–488.

[3] A. Algaba, F. Fernández-Sánchez, M. Merino, A.J. Rodríguez-Luis, Analysis of the T-point-Hopf bifurcation in the Lorenz system. Commun. Nonlinear Sci. Numer. Simul. 22 (2015) 676–691.