A comparison of parametrizations of invariant manifolds for nonlinear model reduction

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Abstract. A comparison of three different parametrization styles for the approximation of invariant manifolds is performed for eight different dynamical systems. The impact of the parametrization style on finite order approximations is investigated by comparison to exact solutions. A modified parametrization style is shown to yield superior results for two example systems thus motivating further research into the development of better parametrizations in the future.

Introduction

The approximation of invariant manifolds is a popular approach for reduced order modeling of nonlinear dynamical systems. Stable and sufficiently attractive (slow) invariant manifolds represent the system's behavior of interest since solution trajectories of the full system that start close to such a manifold converge towards it and may be approximated by the solution trajectory on the invariant manifold. For a given (smooth) dynamical system, invariant manifolds in a neighborhood of a solution such as a fixed point can be approximated numerically by the parametrization method [1, 2]. An ansatz of multivariate polynomials for the invariant manifold and the reduced dynamics thereon yields a sequence of systems of linear equations to determine the unknown coefficients that can be solved recursively starting at the lowest order [3, 4]. However, this procedure gives fewer equations than unknowns which means that there is some methodological ambiguity that needs to be resolved. Depending on what additional conditions are introduced to resolve this ambiguity, this approach gives a certain parametrization—e. g. the graph style parametrization (GSP) or the normal form parametrization (NFP). Different parametrization yield better or worse approximations, depending on the specific dynamic system that they are applied to. Although a posteriori error analysis is possible [5], it is difficult to choose the best parametrization a priori. To illustrate the impact of the parametrization choice and motivate further research into the development of further parametrizations, we apply GSP, NFP and NFP for systems with (near) resonances to a series of benchmark problems that are designed to favor one parametrization style [6].

Results and discussion

Approximations of invariant manifolds and the reduced dynamics thereon are calculated for eight dynamical systems. Exact solution of the invariant manifolds that we want to approximate numerically with all three parametrization styles are known for seven of those systems; the eighth system is a finite element model taken from the literature. Except for the last system, the dynamics of the full systems and their exact invariant manifold are given by multivariate polynomials of order 3 or less. However, most parametrization styles are not able to recover these low dimensional expressions, instead giving infinite series approximations with limited convergence ranges. We present some unexpected results such as one case where a parametrization is able to recover not just the exact invariant manifold, but also another solution, namely a limit cycle on this manifold. Lastly, we show a heuristically modified parametrization gives a superior approximation for the invariant manifold and the reduced dynamics thereon than GSP or NFP at the same order for the eighth system we considered.

The presented study illustrates that the inherent ambiguity, if resolved sub-optimally, gives rise to bad approximations with small convergence ranges. Future developments and improvements of the parametrization method should take this into account. The presented systems may serve as a possible benchmark for any such developments, since their invariant manifolds and solution behavior are known exactly.

References

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