Graph and backward asymptotics of the logistic map

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Abstract. The qualitative behavior of a dynamical system can be encoded into a graph. In this talk we will present our recent results on the graph of the logistic map, joint work with Jim Yorke, and, as an application, our results on its backward asymptotics. Our qualitative results go way beyond 1-dimensional dynamics: our numerical explorations show that the graph of the logistic map appears as subgraph of completely unrelated systems such as the Lorenz system.

Introduction

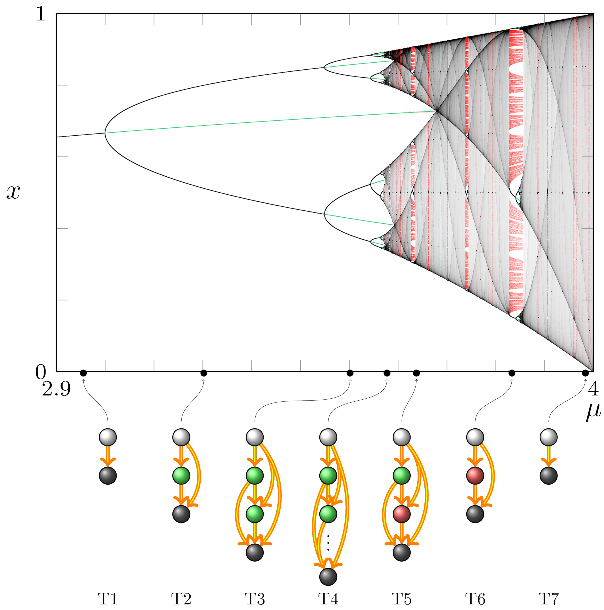
The idea of encoding the qualitative behavior of a Dynamical System (DS) into a graph goes back to Smale [1], that in Sixties introduced it in case of Axiom-A systems. In order to generalize Smale's idea, based on the Non-Wandering set, one needs to replace the Non-Wandering set with the Chain-Recurrent set, introduced in Seventies by Conley [3] in the compact continuous case and later generalized by several other authors to many other settings, including discrete systems. Chain-recurrence is a key ingredient for building the graph of a DS. Given a DS, a point is chain-recurrent when either is periodic or an arbitrarily small amount of control can be added such that the controlled trajectory is a loop passing through p. Points downstream from p are those for which either the trajectory from p goes to q or an arbitrarily small amount of control can be added such that the controlled trajectory goes from p to q. We now extend the stream analogy. If p is downstream from q and q is downstream from p, then we say p and q are in the same pond. The nodes of the graph are precisely these ponds. We say that there is an edge from node A to node B if, arbitrarily close to A, there are points asymptoting to B.

Results and discussion

In [4], we studied the graph the logistic map

$$\ell_\mu(x) = \mu x(1-x).$$

For $\mu \in [0,4]$, ℓ_μ maps $[0,1]$ into itself and has a single attractor. For $\mu \in (2,4)$, its graph has at least two nodes. Our main result is that the nodes of ℓ_μ can be sorted in a linear order $N_0, N_1, \ldots, N_p$, where possibly $p = \infty$, so that there is an edge from node N_i to node N_j if and only if $j > i$ [4]. In Fig. 1 we show the bifurcation diagram of the logistic map for $\mu \in [2.9, 4]$. The diagram shows attracting nodes in gray, repelling periodic orbit nodes in green and repelling Cantor set nodes in red. Below the diagram are shown some example of graphs. The upper (white) node N_0 in all graphs is the fixed endpoint $x = 0$, the lowest (gray) node N_p is the attractor. For instance, case T4 is the graph corresponding to the Feigenbaum parameter $\mu \simeq 3.57$, in which case there are infinitely many nodes, and case T6 is the simplest graph of the logistic map within the period-3 window, that was first studied by Smale and Williams in [2]. We will also show that, correspondingly to the structure of the graph, every point close enough to a node N_i has backward orbits asymptoting to each N_j for $j \leq i$ and to no other node [6]. Finally, we will show numerical results suggesting that our results on the logistic map apply also to much more general higher-dimensional systems [5].



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