Exact dynamical solution of the Kuramoto–Sakaguchi Model for finite networks of identical oscillators

Antonio Mihara* and **Rene O. Medrano-T.****,

*Departamento de Física, Universidade Federal de São Paulo, UNIFESP, Campus Diadema, São Paulo, Brasil

**Departamento de Física, Universidade Federal de São Paulo, UNIFESP, Campus Diadema, São Paulo, Brasil, and Departamento de Física, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, UNESP, Campus Rio Claro, São Paulo, Brasil, ORCID 0000-0003-0866-2466

Abstract. We study the Kuramoto–Sakaguchi model composed by any N identical phase oscillators symmetrically coupled. Ranging from local (one-to-one, $R = 1$) to global (all-to-all, $R = N/2$) couplings, we derive the general solution that describes the network dynamics next to an equilibrium. Therewith we build stability diagrams according to N and R bringing to the light a rich scenery of attractors, repellers, saddles, and non-hyperbolic equilibriums.

Introduction

For more than forty years, the paradigmatic system of N one-dimensional coupled phase oscillators, the Kuramoto model [1], has been intensively studied to understand phenomena related to synchronization in biological, chemical, and electronic networks. Despite the simplicity of the dynamics of each oscillator ($\dot{\theta} = \omega$) strong efforts should be dedicated to find analytical solutions for a network of nonlinearly coupled oscillators, due to the high dimensionality of the system. Kuramoto showed a seminal solution giving rise to the prosper application of the mean–field theory where $N \to \infty$ with oscillators globally coupled. In contrast, accurate results for the finite-size Kuramoto model remains a challenge due to the great number of equations involved, nevertheless, the dynamics is richer. While in the global coupling the full synchronization is the only stable equilibrium, in different topologies of the Kuramoto model multistability is allowed [2]. And, sustained by Lyapunov function argument, the system would reach an equilibrium state as $t \to \infty$ [3].

Multistability, basin of attractions, and traveling waves are some of fundamental phenomena directly related with equilibriums in variants of the Kuramoto model with both attractive and repulsive phase couplings, where the oscillators do not collapse in a single phase although they synchronize in frequency. These phenomena are also observed in real-world networks [4, 5, 6, 7]. Such manifestations are mostly studied in the continuous thermodynamic limit and keep not yet well understood. Exact solutions for lower number of oscillators in the Kuramoto model are mandatory in this study but they are still a topic of investigation.

Results and discussion

In order to shed some light on those problems we study the Kuramoto–Sakaguchi (KS) model [8], a generalization of the Kuramoto model, explicitly for a finite number of N identical oscillators symmetrically coupled ($G = G^T$, in matrix representation). In opposition of traditional investigations, where the time evolution of the network is followed by the order parameter, we obtain solutions describing precisely the individual trajectories of each oscillator when the system is close to an equilibrium. We present several numerical studies in a great accordance with our theoretical predictions and focus in the role of non-hyperbolic equilibria in the general dynamic behavior. More specifically, we determine the set of eigenvalues associated to each state identifying the complete stability scenario of hyperbolic and non-hyperbolic equilibria for a finite number N of oscillators and calculate the bifurcation these states in the thermodynamic limit.

References

[1] Y. Kuramoto, Self-entrainment of a population of coupled non-linear oscillators. In: H. Arakai (Ed.), International Symposium on Mathematical Problems in Theoretical Physics, Lecture Notes in Physics, vol. 39, Springer, New York, p. 420, 1975.

[2] D. A. Wiley and S. H. Strogatz (2006) The size of the sync basin, *Chaos* **16**: 015103.

[3] J. L. Hemmen and W. F. Wreszinski (1993) Lyapunov function for the kuramoto model of nonlinearly coupled oscillators. *J. of Stat. Phys.* **72**: 145.

[4] A. H. Cohen, G. B. Ermentrout, T. Kiemel, N. Kopell, K. A. Sigvardte, and T. L. Williams (1992) Modelling of intersegmental coordination in the lamprey central pattern generator for locomotion. *Trends Neurosci.* **15**: 434.

[5] G. B. Ermentrout and N. Kopell (1994) Inhibition-produced patterning in chains of coupled nonlinear oscillators. *SIAM J. Appl. Math.* **54**: 478.

[6] M. Tsodyks, T. Kenet, A. Grinvald, and A. Arieli (1999) Linking spontaneous activity of single cortical neurons and the underlying functional architecture. *Science* **286**: 1943.

[7] J. P. Newman and R. J. Butera (2010) Mechanism, dynamics, and biological existence of multistability in a large class of bursting neurons. *Chaos* **20**: 023118.

[8] H. Sakaguchi and Y. Kuramoto (1986) A soluble active rotator model showing phase transitions via mutual entrainment. *Prog. Theor. Phys.* **76**: 576.