Implicit Milstein schemes: properties preservation when solving the CIR equation

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Abstract. In this work, the stochastic θ-Milstein method is used to numerically solve the CIR equation. Then, an analysis is conducted on the preservation of the CIR processes properties by the numerical solution, namely, its positivity and its reversion to the long-term mean.

Introduction

The Cox-Ingersoll-Ross (CIR) model describes the interest rate as the solution to the nonlinear equation

$$dX(t) = \alpha(\mu - X(t))dt + \sigma\sqrt{X(t)}dW_t \tag{1}$$

where W_t is a standard Wiener process and $\alpha, \mu, \sigma \in \mathbb{R}^+$. Although the diffusion coefficient does not fulfill the Lipschitz condition, specific results showing the existence and uniqueness of the strong solution can be found in the literature [9]. We are interested in two properties of the exact solution:

(P) Positivity: If $2\alpha\mu > \sigma^2$, the solution remains positive if it starts positive: $X_t > 0$ for $t \in \mathbb{R}^+$ if $X_0 > 0$.

(MR) Mean reversion: The long term mean coincides with the parameter μ: $\lim_{t\to\infty} \mathbb{E}[X(t)] = \mu$.

Numerical methods specially designed to solve the CIR equation have been proposed in the literature [1, 2, 3, 4]. A desirable property of any numerical method for solving an SDE is the preservation of qualitative properties of the exact solution [5]. Our goal is to propose schemes that, applied to the CIR problem, give numerical solutions that preserve properties (P) and (MR).

Results and discussion

We prove that the numerical solution given by the stochastic θ-Milstein methods

$$X_{n+1} = X_n + \alpha(\mu - X_n)\Delta(1-\theta) + \alpha(\mu - X_{n+1})\Delta\theta + \sigma\sqrt{X_n}\Delta W_n + \frac{\sigma^2}{4}\left(\Delta W_n^2 - \Delta\right),$$

with $\theta \geq 1$ to solve the CIR equation (1), preserve the positivity of the exact solution, as well as, without any additional restriction on the step size Δ, the long-term mean of the exact solution. These theoretical results are illustrated on the left and right pictures respectively of Figure 1.

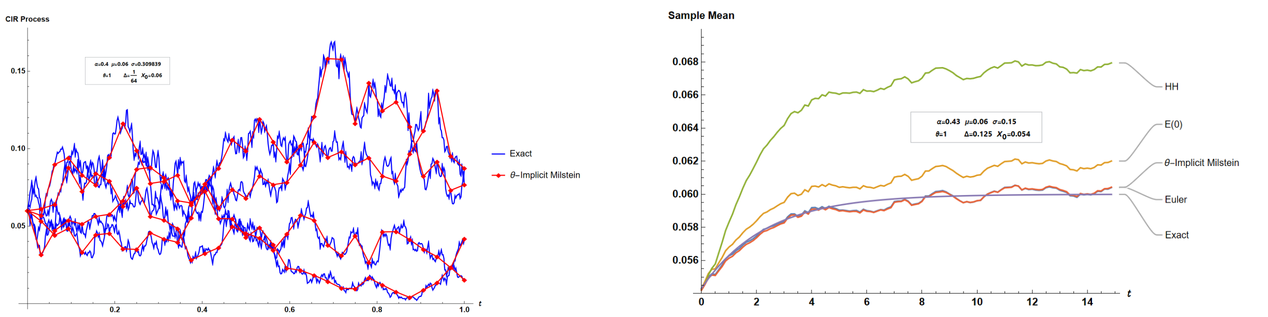

Figure 1: Left: Four trajectories of the exact solution and the corresponding numerical approximations with the fully implicit ($\theta = 1$) Milstein method. Right: Evolution of first moment of the exact solution and the numerical approximations with Euler, θ-Milstein together with the methods proposed in [1] and [4].

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