A quantitative Birkhoff Normal Form for geometrically nonlinear hinged-hinged beams

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Abstract. The study of internal resonances of a system is crucial to investigate its stability. KAM (Kolmogorov Arnold Moser) theory is a powerful branch of perturbation theory born to face the small divisors (resonances) problem in hamiltonian dynamical systems. Its applicability to concrete physical problems is a well-known challenge because of the extreme smallness required for the perturbation parameter. Here we consider an undamped nonlinear hinged-hinged beam with stretching nonlinearity as an infinite dimensional hamiltonian system. We obtain analytically a quantitative Birkhoff Normal Form, via a nonlinear coordinate transformation that allows us to integrate the system up to a small reminder, providing a very precise description of small amplitude solutions over large time scales. The optimization of the involved estimates yields results obtained for realistic values of the physical quantities and of the perturbation parameter.

Introduction

We consider the dimensionless nonlinear beam equation with stretching nonlinearity

$$u_{tt} + u_{xxxx} - \left(m + \frac{1}{2\pi} \int_0^\pi u_x^2 \, dx \right) u_{xx} = 0 \,, \tag{1}$$

for $t \in \mathbb{R}$ and $x \in [0, \pi]$, with the following hinged-hinged boundary conditions: $u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0$. Here $\sqrt{I/A}u$ is the vertical displacement and $m = \frac{L^2 P}{\pi^2 EI}$ indicates the nondimensional axial force, where L, I, A, E, P are, respectively, the length of the beam, the moment of inertia, the cross-section area, the Young modulus, and the tensile axial force (possibly also negative entailing compressive force). Being conservative, Eq. (1) has a hamiltonian structure. Indeed, by letting $\omega_j^2 := j^4 + mj^2$ and $\phi_j(x) := \sqrt{2/\pi} \sin jx$, respectively, denote the eigenvalues and the eigenfunctions of the Sturm–Liouville operator $(\partial_{xxxx} - m\partial_{xx})$ on $[0, \pi]$, the Hamiltonian can be expressed as

$$H(\mathbf{p}, \mathbf{q}) = \textstyle\sum_{j\geq 1} \omega_j I_j + \frac{1}{8\pi} \Big(\sum_{j\geq 1} \frac{j^2}{\omega_j} q_j^2 \Big)^2 \,, \qquad I_j := \frac{1}{2}(p_j^2 + q_j^2) \,, \tag{2}$$

with $\mathbf{q} = (q_1, q_2, \ldots,)$, $\mathbf{p} = (p_1, p_2, \ldots)$ spanning a suitable Hilbert space of sequencies. Then, given a smooth solution $t \to \big(\mathbf{p}(t), \mathbf{q}(t)\big)$ of Hamilton's equations, one finds that $u(t, x) := \sum_{j\geq 1} \frac{q_j(t)}{\sqrt{\omega_j}} \phi_j(x)$ is a solution of (1). If the linear frequencies ω_j are non-resonant, after a canonical change of variables close to the origin, the Hamiltonian is in BNF (Birkhoff Normal Form) up to some order $2d > 0$, namely $H = N(I) + R(\mathbf{p}, \mathbf{q})$, where $N(I) = \sum_{j\geq 1} \omega_j I_j + g(I)$ for some polynomial g of degree d in I and $R = O(|(\mathbf{p}, \mathbf{q})|^{2d+2})$. Since the nonlinear term N is *integrable, the BNF allows a precise description of the solutions with initial data* $\epsilon := |\mathbf{p}(0)| + |\mathbf{q}(0)| \leq \epsilon_0$ *up to times* $|t| \leq T_0 \epsilon^{-2d}$, for suitably small ϵ_0 and T_0. This immediately reads as a stability result for Eq. (1) with ϵ-small initial data $u(0, x)$ and $u_t(0, x)$. There are some results (see, e.g. [1]) on the BNF for the beam equation (with nonlinearities different from (1)) but with no physical applications. Indeed the typical problem in hamiltonian perturbation theory (especially for PDEs) is that the amplitude threshold, ϵ_0 here, is very small.

Results and discussion

For $m > -1$ we prove that the frequencies are non-resonant up to order 4, which, in general, corresponds to show that $\omega_i \pm \omega_j \pm \omega_k \pm \omega_\ell$ does not vanish for suitable combinations of positive integers i, j, k, ℓ and $\pm$ signs. However, in the present case, due to the special form of the nonlinearity, the non-resonance condition reduces to $\omega_i - \omega_j \geq c > 0$ uniformly in $i > j \geq 1$. Then we can put the system in BNF up to order $2d = 4$. Moreover, for $0 < |m| < 1$ we are able to show that the frequencies are non-resonant up to order 6 (which reduces to $|\omega_i - \omega_j - \omega_k| \geq c > 0$ uniformly in $i > j \geq k \geq 1$), so that we can put the system in BNF up to order $2d = 6$. The main point here is that, *by optimizing the estimates, we are able to find realistic values for* ϵ_0 *and* T_0. For example for a steel beam of length $L = 2\text{m}$, height 0.02m, $P = -16.5\text{kN}$ and initial vertical displacement $2 \cdot 10^{-4}\text{m}$ (corresponding to $m = -0.5$, $\epsilon = 0.04$) we have stability time length of 250s (18500 oscillations). As far as we know, this is the first purely analytical result of this kind in hamiltonian perturbation theory for PDEs.

References

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