Critical Dynamics of Kuramoto Model on Erdős-Rényi Random Graphs

Hai Chen*

Department of Physics, Zhejiang University, Hangzhou, China

Abstract. The Kuramoto model(KM) describes the dynamics and synchronization of coupled oscillators and has been intensively investigated in various physical and biological systems. In his original work, Kuramoto studied the fully connected oscillator system and determined its critical coupling strength K_c of phase transition from incoherent initial state to synchronized final state. The complexity of the model increases significantly when the system is on random graphs. In this work, we focus on the Kuramoto model under the Erdős-Rényi random graph topology and study its critical behaviors. Specifically, we demonstrate that statistically there exists an effective critical coupling K_{ec} which is the product of original coupling strength K and the graph link probability p on the two-dimensional critical curve. Under one-dimensional projection, the critical link probability p_c is inversely proportional to the coupling strength K and vice versa. We generate a large numerical data sample to simulate the KM on this topology and obtain well agreement between the semi-quantitative analysis and simulation result. These results provide insights on Kuramoto model's critical behavior on random graphs, and can be applied to determine real dynamical system's intrinsic properties and extended to other stochastic topologies.

Introduction

Synchronization of interacting elements is ubiquitous in nature, and has been widely investigated in many physical and biological systems, such as flashing fireflies, neurons in the brain, electric power grids and Josephson junction arrays[1]. Kuramoto introduced an analytically solvable model of coupled oscillators, and thus inspired extensive studies on phase synchronization research since the 1980s[2, 3]. In spite of its mature age, the theory of synchronization is still full of surprises, applications, and new features. The synchronization of coupled oscillators depends on many factors, such as the coupling strength, the network topology, the natural frequency distribution, interaction time delay, etc[4]. One of the key factors, the network topology, determines how interaction and information propagate among the elements. As the network topology's complexity increases, the evaluation of Kuramoto model's critical dynamics becomes very challenging[5]. The random graph model proposed by Paul Erdős and Alfred Rényi is simple yet elegant, and can be integrated into the Kuramoto model by connecting each pair of oscillators with a fixed link probability p. In spite of its limitations in simulating real-world networks that follow power-law degree distribution, it is the basis of many variations of random graph models, and has important applications in statistical physics(e.g., percolation theory[6]).

Summary

With statistical analysis and extensive numerical simulation, the critical dynamics of Kuramoto model on Erdős–Rényi random graph have been resolved in this work. We first show that under ER topology it is statistically equivalent to the traditional fully connected form, with effective coupling strength K_e reduced according to the link probability p. The two-dimensional critical curve of phase transition reveals the simple inverse proportional relations between link probability p and coupling strength K. This result agrees with the rigorous mathematical calculation by Chiba and Medvedev[7, 8]. Meanwhile this leads to possibility of probing a physical system's intrinsic coupling strength if it follows ER random graphs topology. On the basis of this work, it is possible to investigate the critical behaviors under more complex network topology, such as small-world and other stochastic graphs. We will also extend the topology studies with the presence of noise, time-delay, inertia, etc., to explore new features of Kuramoto model.

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